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Ornstein-Uhlenbeck processes driven by cylindrical Lévy processes

Markus Riedle*
Department of Mathematics
King's College
London WC2R 2LS
United Kingdom

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Abstract

In this article we introduce a theory of integration for deterministic, operator-valued integrands with respect to cylindrical Lévy processes in separable Banach spaces. Here, a cylindrical Lévy process is understood in the classical framework of cylindrical random variables and cylindrical measures, and thus, it can be considered as a natural generalisation of cylindrical Wiener processes or white noises. Depending on the underlying Banach space, we provide necessary and/or sufficient conditions for a function to be integrable. In the last part, the developed theory is applied to define Ornstein-Uhlenbeck processes driven by cylindrical Lévy processes and several examples are considered.

1 Introduction

The degree of freedom of models in infinite dimensions is often reflected by the constraint that each mode along a one-dimensional subspace is independently perturbed by the noise. In the Gaussian setting, this leads to the *cylindrical Wiener process* including from a modeling point of view the very important possibility to describe a Gaussian noise in both time and space with a great flexibility, i.e. space-time white noise. Up to very recently, there has been no analogue for Lévy processes. The notion *cylindrical Lévy process* appears the first time in the monograph [26] by Peszat and Zabczyk and it is followed by the works Brzeźniak et al [6], Brzeźniak and Zabczyk [8], Liu and Zhai [18], Peszat and Zabczyk [27] and Priola and Zabczyk [28]. The first systematic introduction of cylindrical Lévy processes appears in our work Applebaum and Riedle [1]. In this work cylindrical Lévy processes are introduced as a natural generalisation of cylindrical Wiener processes, and they model a very general, discontinuous noise occurring in the time and state space.

The aforementioned literature ([6], [8], [18], [27], [28]) study stochastic evolution equations of the form

$$dY(t) = AY(t) dt + dL(t) \quad \text{for all } t \geq 0, \quad (1.1)$$

where A is the generator of a strongly continuous semigroup on a Hilbert or Banach space. The driving noise L differs in these publications but it is always constructed in an explicit

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way and it is referred to by the authors as *Lévy white noise*, *cylindrical stable process* or just *Lévy noise*. The works have in common that the solution of (1.1) is represented by a stochastic convolution integral, which is based either on the one-dimensional integration theory, if the setting allows as for instance in [6], [27], or on moment inequalities for Poisson random measures as for instance in [8]. However, these approaches and the results are tailored to the specific kind of noise under consideration, respectively.

The main objective of our work is to develop a general theory of stochastic integration for deterministic integrands, which provides a unified framework for the aforementioned works. Although not part of this work, the results are expected to lead to a better understanding of phenomena, which are individually observed for the solutions of (1.1) in the various models considered in the literature, such as irregularity of trajectories in [6]. In order to be able to develop a general theory, we define cylindrical Lévy processes by following the classical approach to cylindrical measures and cylindrical processes, which is presented for example in Badrikian [2] or Schwartz [34]. This systematic approach for cylindrical Lévy processes is developed in our work together with Applebaum in [1]. In the current work, we illustrate that those kinds of cylindrical Lévy noise, considered in the literature, are specific examples of a cylindrical Lévy process in our approach.

Integration for random integrands with respect to other cylindrical processes than the cylindrical Wiener process is only considered in a few works. In fact, we are only aware of two approaches to integration with respect to cylindrical martingales, which originate either in the work developed by Métivier and Pellaumail in [20] and [21] or by Mikulevičius and Rozovskii in [22] and [23]. However, both constructions heavily rely on the assumption of finite weak second moments and are therefore not applicable in our framework. For cylindrical Lévy processes with weak second moments, a straightforward integration theory is introduced in Riedle [31]. It is worth mentioning that a localising procedure cannot be applied to cylindrical Lévy processes since they do not necessarily attain values in the underlying space.

Our work can be seen as a generalisation of the publications by Chojnowska-Michalik [9] on the one hand and by Brzeźniak and van Neerven [7] and by van Neerven and Weis [36] on the other hand. In [9] the author introduces a stochastic integral for deterministic integrands with respect to genuine Lévy processes in Hilbert spaces. If we apply our approach to this specific setting, the class of admissible integrands for the integral, developed in our work, is much larger than the one in [9], see Remark 5.11. The articles [7] and [36] introduce a stochastic integral with respect to a cylindrical Wiener processes in Banach spaces. But the approach in [7] and [36] differs from ours as the Gaussian distribution enables the authors to rely on an isometry in terms of square means.

Our approach to develop a stochastic integral is based on the idea to introduce first a cylindrical integral, which exists under mild conditions, since cylindrical random variables are more general objects than genuine random variables. A function is then called stochastically integrable if its cylindrical integral is actually induced by a genuine random variable in the underlying Banach space, which is then called the stochastic integral. The advantage of this approach is that the latter step is a purely measure-theoretical problem, which can be formulated in terms of the characteristic function of the cylindrical integral. Since the stochastic integral, if it exists, must be infinitely divisible, we can describe the class of integrable functions by using conditions which guarantee the existence of an infinitely divisible random variable for a given semimartingale characteristics. The candidate for the semimartingale characteristics is provided by the cylindrical integral, since its characteristic function coincides with the one of its possible extension to a genuine random variable. Conditions, guaranteeing the existence of an infinitely divisible measure in terms of the characteristic function, are known in many spaces, e.g. in Hilbert spaces and in Banach

spaces of Rademacher type. The developed theory of stochastic integration is applied to define Ornstein-Uhlenbeck processes driven by cylindrical Lévy processes. We show in two corollaries, that our approach easily recovers results from the literature on the existence of Ornstein-Uhlenbeck processes.

The organisation of this paper is as follows. In Section 2 we collect some preliminaries on cylindrical measures and cylindrical random variables, which can be found for example in Badrikian [2] and Schwartz [34]. In Section 3 we recall the definition of a cylindrical Lévy process, based on our paper [1] with Applebaum, and we cite some results on its characteristics from Riedle [30]. Furthermore, some important properties of cylindrical Lévy processes are established. Section 4 provides several examples of cylindrical Lévy processes. In particular, we show that the noises, considered in the aforementioned publications, are covered by our systematic approach. In Section 5 we develop a theory of stochastic integration for deterministic, operator-valued integrands with respect to cylindrical Lévy processes. We finish this work with Section 6 where we apply the developed integration theory to treat Ornstein-Uhlenbeck processes.

2 Preliminaries

Let U be a separable Banach space with dual U^* . The dual pairing is denoted by $\langle u, u^* \rangle$ for $u \in U$ and $u^* \in U^*$. The Borel σ -algebra in U is denoted by $\mathcal{B}(U)$ and the closed unit ball at the origin by $B_U := \{u \in U : \|u\| \leq 1\}$. The space of positive, finite Borel measures on $\mathcal{B}(U)$ is denoted by $M(U)$ and it is equipped with the topology of weak convergence. The Bochner space is denoted by $L^1([0, T]; U)$ and it is equipped with the standard norm.

For every $u_1^*, \dots, u_n^* \in U^*$ and $n \in \mathbb{N}$ we define a linear map

$$\pi_{u_1^*, \dots, u_n^*} : U \rightarrow \mathbb{R}^n, \quad \pi_{u_1^*, \dots, u_n^*}(u) = (\langle u, u_1^* \rangle, \dots, \langle u, u_n^* \rangle).$$

Let Γ be a subset of U^* . Sets of the form

$$\begin{aligned} C(u_1^*, \dots, u_n^*; B) &:= \{u \in U : (\langle u, u_1^* \rangle, \dots, \langle u, u_n^* \rangle) \in B\} \\ &= \pi_{u_1^*, \dots, u_n^*}^{-1}(B), \end{aligned}$$

where $u_1^*, \dots, u_n^* \in \Gamma$ and $B \in \mathcal{B}(\mathbb{R}^n)$ are called *cylindrical sets*. The set of all cylindrical sets is denoted by $\mathcal{Z}(U, \Gamma)$ and it is an algebra. The generated σ -algebra is denoted by $\hat{\mathcal{Z}}(U, \Gamma)$ and it is called the *cylindrical σ -algebra with respect to (U, Γ)* . If $\Gamma = U^*$ we write $\mathcal{Z}(U) := \mathcal{Z}(U, \Gamma)$ and $\hat{\mathcal{Z}}(U) := \hat{\mathcal{Z}}(U, \Gamma)$.

A function $\eta : \mathcal{Z}(U) \rightarrow [0, \infty]$ is called a *cylindrical measure on $\mathcal{Z}(U)$* , if for each finite subset $\Gamma \subseteq U^*$ the restriction of η to the σ -algebra $\hat{\mathcal{Z}}(U, \Gamma)$ is a measure. A cylindrical measure η is called finite if $\eta(U) < \infty$ and a cylindrical probability measure if $\eta(U) = 1$.

For every function $f : U \rightarrow \mathbb{C}$ which is measurable with respect to $\hat{\mathcal{Z}}(U, \Gamma)$ for a finite subset $\Gamma \subseteq U^*$ the integral $\int f(u) \eta(du)$ is well defined as a complex valued Lebesgue integral if it exists. In particular, the characteristic function $\varphi_\eta : U^* \rightarrow \mathbb{C}$ of a finite cylindrical measure η is defined by

$$\varphi_\eta(u^*) := \int_U e^{i\langle u, u^* \rangle} \eta(du) \quad \text{for all } u^* \in U^*.$$

Let (Ω, \mathcal{A}, P) be a probability space. The space of equivalence classes of measurable functions $f : \Omega \rightarrow U$ is denoted by $L_P^0(\Omega; U)$ and it is equipped with the topology of convergence in probability.

Similarly to the correspondence between measures and random variables there is an analogous random object associated to cylindrical measures: a *cylindrical random variable* Z in U is a linear and continuous map

$$Z: U^* \rightarrow L_P^0(\Omega; \mathbb{R}).$$

Here, continuity is with respect to the norm topology on U^* and the topology of convergence in probability. A family $(Z(t) : t \geq 0)$ of cylindrical random variables $Z(t)$ is called a *cylindrical process*. The characteristic function of a cylindrical random variable Z is defined by

$$\varphi_Z: U^* \rightarrow \mathbb{C}, \quad \varphi_Z(u^*) = E[\exp(iZu^*)].$$

If $C = C(u_1^*, \dots, u_n^*; B)$ is a cylindrical set for $u_1^*, \dots, u_n^* \in U^*$ and $B \in \mathcal{B}(\mathbb{R}^n)$ we obtain a cylindrical probability measure η by the prescription

$$\eta(C) := P((Zu_1^*, \dots, Zu_n^*) \in B).$$

We call η the *cylindrical distribution* of Z and the characteristic functions φ_η and φ_Z of η and Z coincide. Conversely, for every cylindrical probability measure η on $\mathcal{Z}(U)$ there exist a probability space (Ω, \mathcal{A}, P) and a cylindrical random variable $Z: U^* \rightarrow L_P^0(\Omega; \mathbb{R})$ such that η is the cylindrical distribution of Z , see [35, VI.3.2].

Let ϑ be an infinitely divisible probability measure on $\mathcal{B}(U)$. Then the characteristic function $\varphi_\vartheta: U^* \rightarrow \mathbb{C}$ of ϑ is given for each $u^* \in U^*$ by

$$\varphi_\vartheta(u^*) = \exp \left(i\langle b, u^* \rangle - \frac{1}{2} \langle Ru^*, u^* \rangle + \int_U \left(e^{i\langle u, u^* \rangle} - 1 - i\langle u, u^* \rangle \mathbb{1}_{B_U}(u) \right) \nu(du) \right), \quad (2.1)$$

where $b \in U$, $R: U^* \rightarrow U$ is the covariance operator of a Gaussian measure on $\mathcal{B}(U)$ and ν is a σ -finite measure on $\mathcal{B}(U)$. Since the triplet (b, R, ν) is unique ([17, Th.5.7.3]), it characterises the distribution of the probability measure ϑ , and it is called the *characteristics* of ϑ . If X is an U -valued random variable which is infinitely divisible, then we call the characteristics of its probability distribution the characteristics of X .

In general Banach spaces it is not as straightforward to define a Lévy measure as in Hilbert spaces. In this work we use the following result (Theorem 5.4.8 in Linde [12]) as the definition: a σ -finite measure ν on a Banach space U is called a *Lévy measure* if

- (i) $\int_U (\langle u, u^* \rangle^2 \wedge 1) \nu(du) < \infty$ for all $u^* \in U^*$;
- (ii) there exists a measure on $\mathcal{B}(U)$ with characteristic function

$$\varphi(u^*) = \exp \left(\int_U \left(e^{i\langle u, u^* \rangle} - 1 - i\langle u, u^* \rangle \mathbb{1}_{B_U}(u) \right) \nu(du) \right).$$

Let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration for the probability space (Ω, \mathcal{A}, P) . An adapted, stochastic process $L := (L(t) : t \geq 0)$ with values in U is called a *Lévy process* if $L(0) = 0$ P -a.s., L has independent and stationary increments and L is continuous in probability. It follows that there exists a version of L with paths which are continuous from the right and have limits from the left (càdlàg paths). The random variable $L(1)$ is infinitely divisible and we call its characteristics the characteristics of L .

3 Cylindrical Lévy processes

Let U be a separable Banach space. A cylindrical probability measure η on $\mathcal{Z}(U)$ is called *infinitely divisible* if for each $k \in \mathbb{N}$ there exists a cylindrical probability measure η_k such that $\eta = (\eta_k)^{*k}$. In [1] and [30] we show that the characteristic function $\varphi_\eta: U^* \rightarrow \mathbb{C}$ of η can be represented by

$$\begin{aligned} \varphi_\eta(u^*) &= \exp \left(ia(u^*) - \frac{1}{2}qu^* + \int_U \left(e^{i\langle u, u^* \rangle} - 1 - i\langle u, u^* \rangle \mathbb{1}_{B_{\mathbb{R}}}(\langle u, u^* \rangle) \right) \mu(du) \right) \\ &=: \exp(\Psi(u^*)), \end{aligned} \quad (3.1)$$

where $a: U^* \rightarrow \mathbb{R}$ is a mapping with $a(0) = 0$ and which is continuous on finite dimensional subspaces, $q: U^* \rightarrow \mathbb{R}$ is a quadratic form and μ is a cylindrical measure on $\mathcal{Z}(U)$ satisfying

$$\int_U (\langle u, u^* \rangle^2 \wedge 1) \mu(du) < \infty \quad \text{for all } u^* \in U^*. \quad (3.2)$$

Consequently, the triplet (a, q, μ) characterises the distribution of the cylindrical measure η and thus, it is called the *(cylindrical) characteristics of η* . The mapping $\Psi: U^* \rightarrow \mathbb{C}$ is called the *(cylindrical) symbol of η* .

We call a cylindrical measure μ on $\mathcal{Z}(U)$ a *(cylindrical) Lévy measure* if it satisfies (3.2). However, note that it is not sufficient for a cylindrical measure μ to satisfy (3.2) in order to guarantee that there exists a corresponding infinitely divisible cylindrical measure with characteristics $(0, 0, \mu)$, see [30]. For a cylindrical or classical Lévy measure μ we denote $\mu^-(C) := \mu(-C)$ for all $C \in \mathcal{Z}(U)$.

A cylindrical process $(L(t) : t \geq 0)$ is called a *cylindrical Lévy process in U* if for all $u_1^*, \dots, u_n^* \in U^*$ and $n \in \mathbb{N}$ we have that

$$((L(t)u_1^*, \dots, L(t)u_n^*) : t \geq 0)$$

is a Lévy process in \mathbb{R}^n . This definition is introduced in our work [1]. It follows that the cylindrical distribution η of $L(1)$ is infinitely divisible and that the characteristic function of $L(t)$ for all $t \geq 0$ is given by

$$\varphi_{L(t)}: U^* \rightarrow \mathbb{C}, \quad \varphi_{L(t)}(u^*) = \exp(t\Psi(u^*)), \quad (3.3)$$

where $\Psi: U^* \rightarrow \mathbb{C}$ is the symbol of η . We call the symbol Ψ and the characteristics (a, q, μ) of η the *(cylindrical) symbol* and the *(cylindrical) characteristics of L* .

A cylindrical Lévy process $(L(t) : t \geq 0)$ with characteristics (a, q, μ) can be decomposed into

$$L(t) = W(t) + P(t) \quad \text{for all } t \geq 0, \quad (3.4)$$

where $W(t)$ and $P(t)$ are linear maps from U^* to $L_P^0(\Omega; \mathbb{R})$, see [1, Th.3.9]. For each $u^* \in U^*$ the stochastic processes $(W(t)u^* : t \geq 0)$ and $(P(t)u^* : t \geq 0)$ are unique up to indistinguishability by [15, Th.I.4.18]. In addition to the assumed continuity of the operator $L(t)$ we require in this work that $W(t)$ and $P(t)$ are continuous for each $t \geq 0$, in which case $(W(t) : t \geq 0)$ is a cylindrical Lévy process with characteristics $(0, q, 0)$ and $(P(t) : t \geq 0)$ is an independent, cylindrical Lévy process with characteristics $(a, 0, \mu)$. Lemma 4.4. in [30] guarantees that the characteristics (a, q, μ) obeys:

- (1) $a: U^* \rightarrow \mathbb{R}$ is continuous;

(2) there exists a positive, symmetric operator $Q: U^* \rightarrow U^{**}$ such that

$$qu^* = \langle u^*, Qu^* \rangle \text{ for all } u^* \in U^*;$$

(3) for every sequence $(u_n^*)_{n \in \mathbb{N}} \subseteq U^*$ with $\|u_n^* - u_0^*\| \rightarrow 0$ for some $u_0^* \in U^*$ it follows that

$$(|\beta|^2 \wedge 1)(\mu \circ (u_n^*)^{-1})(d\beta) \rightarrow (|\beta|^2 \wedge 1)(\mu \circ (u_0^*)^{-1})(d\beta) \text{ weakly in } M(\mathbb{R}). \quad (3.5)$$

In this case, we replace the covariance q by the covariance operator Q and write (a, Q, μ) for the cylindrical characteristics.

We will need several times the following property of an arbitrary cylindrical Lévy measure in a Banach space. For classical Lévy measures, the same property can be deduced by different arguments, see [17, Pro.5.4.5].

Lemma 3.1. *Let μ be the cylindrical Lévy measure of a cylindrical Lévy process in U . Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\sup_{\|u^*\| \leq \delta} \int_U (|\langle u, u^* \rangle|^2 \wedge 1) \mu(du) \leq \varepsilon.$$

Proof. Due to (3.4) we can assume that the cylindrical Lévy process L has the characteristics $(a, 0, \mu)$. If L' denotes an independent copy of L then the cylindrical Lévy process $\tilde{L} := L - L'$ has the characteristics $(0, 0, \mu + \mu^-)$.

Define for every $u^* \in U^*$ the cylindrical set $D(u^*) := \{u \in U : |\langle u, u^* \rangle| \leq 1\}$. The inequality $1 - \cos(\beta) \geq \frac{1}{3}\beta^2$ for all $|\beta| \leq 1$ implies by using the symmetry of $\mu + \mu^-$, that the characteristic function of $\tilde{L}(1)$ satisfies for each $u^* \in U^*$:

$$\begin{aligned} \varphi_{\tilde{L}(1)}(u^*) &= \exp \left(- \int_U (1 - \cos(\langle u, u^* \rangle)) (\mu + \mu^-)(du) \right) \\ &\leq \exp \left(- \int_{D(u^*)} (1 - \cos(\langle u, u^* \rangle)) (\mu + \mu^-)(du) \right) \\ &\leq \exp \left(- \frac{2}{3} \int_{D(u^*)} |\langle u, u^* \rangle|^2 \mu(du) \right). \end{aligned}$$

Consequently, we obtain

$$\int_{D(u^*)} |\langle u, u^* \rangle|^2 \mu(du) \leq -\frac{3}{2} \ln(\varphi_{\tilde{L}(1)}(u^*)) \quad \text{for all } u^* \in U^*.$$

Since $\tilde{L}(1): U^* \rightarrow L_P^0(\Omega; \mathbb{R})$ is continuous, its characteristic function $\varphi_{\tilde{L}(1)}: U^* \rightarrow \mathbb{R}$ is continuous, see [35, Pro.IV.3.4]. Therefore, there exists a $\delta_1 > 0$ such that

$$\sup_{\|u^*\| \leq \delta_1} \int_{D(u^*)} |\langle u, u^* \rangle|^2 \mu(du) < \varepsilon. \quad (3.6)$$

For the second part of the proof, we define $d(u^*) := \mu(D(u^*)^c)$ for all $u^* \in U^*$, and we show that for every $\varepsilon > 0$ there exists a $\delta_2 > 0$ such that

$$\sup_{\|u^*\| \leq \delta_2} d(u^*) \leq \varepsilon. \quad (3.7)$$

Assume for a contradiction that (3.7) is not satisfied. Then there exists a sequence $(u_n^*)_{n \in \mathbb{N}} \subseteq U^*$ with $u_n^* \rightarrow 0$ as $n \rightarrow \infty$ and $d(u_n^*) > \varepsilon$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ define the stopping time $\tau_{u_n^*} := \inf\{t \geq 0 : |(L(t) - L(t-))u_n^*| > 1\}$. Since for each $n \in \mathbb{N}$ the stopping time $\tau_{u_n^*}$ is exponentially distributed with parameter $d(u_n^*)$ it follows that

$$P\left(\sup_{t \in [0, T]} |L(t)u_n^*| \leq \frac{1}{2}\right) \leq P(\tau_{u_n^*} > T) = e^{-d(u_n^*)T} < e^{-\varepsilon T} \quad \text{for all } n \in \mathbb{N}. \quad (3.8)$$

Let $D([0, T]; \mathbb{R})$ denote the space of functions on $[0, T]$ with càdlàg trajectories and endow this space with the supremum norm. Define the mapping $L: U^* \rightarrow L^0(\Omega; D([0, T]; \mathbb{R}))$ by $Lu^* := (L(t)u^* : t \in [0, T])$. It follows by the closed graph theorem for F -spaces (see [37, Th.II.6.1]), that L is a continuous mapping. Consequently, we obtain $\sup_{t \in [0, T]} L(t)u_n^* \rightarrow 0$ in probability as $n \rightarrow \infty$, which contradicts (3.8). Thus, we have proved equality (3.7), which together with (3.6) completes the proof. \square

The first application of the previous Lemma 3.1 establishes that the cylindrical symbol Ψ maps bounded sets into bounded sets.

Lemma 3.2. *The cylindrical symbol Ψ satisfies for each $c > 0$:*

$$\sup_{\|u^*\| \leq c} |\Psi(u^*)| < \infty.$$

Proof. Let L be a cylindrical Lévy process with symbol Ψ and characteristics (a, Q, μ) , so that Ψ is of the form (3.1). Since $L(1)(\beta u^*)$ and $\beta L(1)u^*$ are identically distributed for every $u^* \in U^*$ and $\beta > 0$, equating their Lévy-Khintchine formula yields

$$a(\beta u^*) = \beta a(u^*) + \beta \int_U \langle u, u^* \rangle \left(\mathbb{1}_{B_{\mathbb{R}}}(\beta \langle u, u^* \rangle) - \mathbb{1}_{B_{\mathbb{R}}}(\langle u, u^* \rangle) \right) \mu(du). \quad (3.9)$$

The second term on the right hand side can be estimated by

$$\begin{aligned} & \int_U |\langle u, u^* \rangle| \left| \mathbb{1}_{B_{\mathbb{R}}}(\beta \langle u, u^* \rangle) - \mathbb{1}_{B_{\mathbb{R}}}(\langle u, u^* \rangle) \right| \mu(du) \\ & \leq \int_{|\langle u, u^* \rangle| \leq \frac{1}{\beta}} \langle u, u^* \rangle^2 \mu(du) + \int_{|\langle u, u^* \rangle| > \frac{1}{\beta}} \mu(du) = \int_U \left(\langle u, u^* \rangle^2 \wedge \frac{1}{\beta^2} \right) \mu(du). \end{aligned} \quad (3.10)$$

The continuity of a and $a(0) = 0$ imply that there exists a $\delta > 0$ such that $|a(u^*)| \leq 1$ for all $\|u^*\| \leq \delta$. By choosing $\beta = \frac{\varepsilon}{\delta}$ it follows from (3.9) and (3.10) by Lemma 3.1, that

$$\sup_{\|u^*\| \leq c} |a(u^*)| \leq \beta \sup_{\|u^*\| \leq c} \left| a\left(\frac{u^*}{\beta}\right) \right| + \beta \sup_{\|u^*\| \leq c} \int_U \left(|\langle u, u^* \rangle|^2 \wedge \frac{1}{\beta^2} \right) \mu(du) < \infty. \quad (3.11)$$

Boundedness of the term in (3.1) involving Q can easily be established since $Q \in \mathcal{L}(U^*, U^{**})$. Applying the estimate

$$\sup_{\|u^*\| \leq c} \int_U \left| e^{i\langle u, u^* \rangle} - 1 - i\langle u, u^* \rangle \mathbb{1}_{B_{\mathbb{R}}}(\langle u, u^* \rangle) \right| \mu(du) \leq 2 \sup_{\|u^*\| \leq c} \int_U \left(|\langle u, u^* \rangle|^2 \wedge 1 \right) \mu(du)$$

completes the proof by another application of Lemma 3.1. \square

It is well known that if the covariance of a Gaussian cylindrical measure is majorised by the covariance of a Gaussian measure, it extends to a measure and is Gaussian. Next we will derive the analogue result for cylindrical Lévy measures, following the presentation of the result in the Gaussian setting:

Theorem 3.3. *Let η be a centralised, Gaussian cylindrical measure on $\mathcal{Z}(U)$ with covariance $q: U^* \rightarrow \mathbb{R}$, i.e.*

$$q(u^*) = \int_U \langle u, u^* \rangle^2 \eta(du).$$

If ϑ is a Gaussian measure on $\mathcal{B}(U)$ with covariance operator $R: U^ \rightarrow U$ satisfying*

$$q(u^*) \leq \langle u^*, Ru^* \rangle \quad \text{for all } u^* \in U^*,$$

then η extends to a measure on $\mathcal{B}(U)$ and the extension is Gaussian.

Proof. See Theorem 3.3.1 in [3]. □

We extend this result to cylindrical Lévy measures by generalising Prokhorov's theorem on projective limits ([4, Th.9.12.2] or [35, Th.VI.3.2]) to σ -finite measures. We follow here the proof of [14] in the form as nicely rewritten in [3].

Theorem 3.4. *Let μ be a cylindrical Lévy measure on $\mathcal{Z}(U)$ and ν be a Lévy measure on $\mathcal{B}(U)$ satisfying $\mu \leq \nu$ on $\mathcal{Z}(U)$. Then μ extends to a σ -finite measure on $\mathcal{B}(U)$ and the extension is a Lévy measure.*

Proof. Fix a $\delta > 0$ and define for each $d \in \mathbb{N}$ the rectangle

$$R_\delta^d := \left\{ (\beta_1, \dots, \beta_d) \in \mathbb{R}^d : \sup_{i=1, \dots, d} |\beta_i| \leq \delta \right\},$$

and $B_\delta := \{u \in U : \|u\| \leq \delta\}$. Since U is separable we can choose a norming sequence $\{u_k^*\}_{k \in \mathbb{N}} \subseteq U^*$ with $\|u_k^*\| = 1$, i.e.

$$\|u\| = \sup_{k \in \mathbb{N}} |\langle u, u_k^* \rangle| \quad \text{for all } u \in U.$$

We define for every $k \in \mathbb{N}$ the mapping $\pi_k: U \rightarrow \mathbb{R}^k$ by $\pi_k(u) := (\langle u, u_1^* \rangle, \dots, \langle u, u_k^* \rangle)$, and the finite measure

$$\mu_\delta^k := \mu \circ \pi_k^{-1}|_{\mathbb{R}^k \setminus R_\delta^k} \quad \text{on } \mathcal{B}(\mathbb{R}^k \setminus R_\delta^k).$$

Since ν is a Lévy measure there exists an increasing sequence $\{K_n\}_{n \in \mathbb{N}}$ of compact sets $K_n \subseteq U$ (see [17, Th.5.4.8]), such that

$$\nu\left(\{u \in U : u \in K_n^c, \|u\| > \delta\}\right) \leq \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

By denoting the constant $c_\delta := \nu(B_\delta^c)$ define the set

$$S_\delta := \left\{ \vartheta \in M(U) : \vartheta(U) \leq c_\delta, \vartheta(K_n^c) \leq \frac{1}{n} \quad \text{for all } n \in \mathbb{N} \right\}.$$

Obviously, the set S_δ is non-empty and relatively compact in $M(U)$ by Prohorov's Theorem; see [35, Th.I.3.6]. Furthermore, for each $k, n \in \mathbb{N}$ we obtain

$$\begin{aligned} \mu_\delta^k((\pi_k(K_n))^c) &= (\mu \circ \pi_k^{-1})((\pi_k(K_n))^c \setminus R_\delta^k) \\ &\leq (\nu \circ \pi_k^{-1})((\pi_k(K_n))^c \setminus R_\delta^k) \\ &= \nu\left(\{u \in U : u \in K_n^c, \sup_{i=1, \dots, k} |\langle u, u_i^* \rangle| > \delta\}\right) \\ &\leq \nu\left(\{u \in U : u \in K_n^c, \|u\| > \delta\}\right) \\ &\leq \frac{1}{n}. \end{aligned}$$

Theorem 9.1.9 in [4] implies that for each $k \in \mathbb{N}$ there exists a measure $\vartheta_\delta^k \in M(U)$ such that

$$\vartheta_\delta^k \circ \pi_k^{-1} = \mu_\delta^k \quad \text{on } \mathcal{B}(\mathbb{R}^k \setminus R_\delta^k),$$

with $\vartheta_\delta^k(U) = \mu_\delta^k(\mathbb{R}^k \setminus R_\delta^k)$ and $\vartheta_\delta^k(K_n^c) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Since $\mu_\delta^k(\mathbb{R}^k \setminus R_\delta^k) \leq c_\delta$, the set

$$S_\delta^k := \{\vartheta \in \bar{S}_\delta : \vartheta \circ \pi_k^{-1} = \mu_\delta^k \text{ on } \mathcal{B}(\mathbb{R}^k \setminus R_\delta^k)\}$$

is non-empty. For $k \leq \ell$ denote by $\pi_{k\ell}$ the natural projection from \mathbb{R}^ℓ to \mathbb{R}^k . Since $\pi_{k\ell}^{-1}(\mathbb{R}^k \setminus R_\delta^k) \subseteq \mathbb{R}^\ell \setminus R_\delta^\ell$ we obtain for $\vartheta \in S_\delta^\ell$ that

$$\vartheta \circ \pi_k^{-1} = (\vartheta \circ \pi_\ell^{-1}) \circ \pi_{k\ell}^{-1} = \mu_\delta^\ell \circ \pi_{k\ell}^{-1} = (\mu \circ \pi_\ell^{-1}) \circ \pi_{k\ell}^{-1} = \mu \circ \pi_k^{-1} \quad \text{on } \mathcal{B}(\mathbb{R}^k \setminus R_\delta^k),$$

which shows $S_\delta^\ell \subseteq S_\delta^k$. Since \bar{S}_δ is compact and S_δ^k is closed for all $k \in \mathbb{N}$, the nested system $\{S_\delta^k : k \in \mathbb{N}\}$ has a non-empty intersection. Thus, for each $\delta > 0$ there exists a measure $\vartheta_\delta \in M(U)$ satisfying

$$\vartheta_\delta \circ \pi_k^{-1} = \mu \circ \pi_k^{-1} \quad \text{on } \mathcal{B}(\mathbb{R}^k \setminus R_\delta^k) \quad \text{for all } k \in \mathbb{N}.$$

The measure ϑ_δ is uniquely determined on $\mathcal{B}(U) \cap B_\delta^c$ since the family of sets

$$\{Z \in \mathcal{Z}(U) : Z = \pi_k^{-1}(B \cap \mathbb{R}^k \setminus R_\delta^k) \text{ for } B \in \mathcal{B}(\mathbb{R}^k), k \in \mathbb{N}\}$$

generates the σ -algebra $\mathcal{B}(U) \cap B_\delta^c$ and it is closed under intersection. Define the measure $\tilde{\vartheta}_{1/k}$ to be the restriction of $\vartheta_{1/k}$ on the disc $\{u \in U : \frac{1}{k} < \|u\| \leq \frac{1}{k-1}\}$ for $k \geq 2$ and $\tilde{\vartheta}_1$ to be the restriction of ϑ_1 to B_1^c . Then

$$\vartheta := \sum_{k=1}^{\infty} \tilde{\vartheta}_{1/k}$$

defines a σ -finite measure ϑ on $\mathcal{B}(U)$ satisfying $\vartheta = \mu$ on $\mathcal{Z}(U)$. Since $\vartheta \leq \nu$ on $\mathcal{Z}(U)$ and $\mathcal{Z}(U)$ is a generator of $\mathcal{B}(U)$ closed under intersection, we have $\vartheta \leq \nu$ on $\mathcal{B}(U)$. Proposition 5.4.5 in [17] guarantees that ϑ is a Lévy measure. \square

4 Examples of cylindrical Lévy processes

Example 4.1. If $(Y(t) : t \geq 0)$ is a genuine Lévy process with values in a Banach space U , then, for $t \geq 0$,

$$L(t) : U^* \rightarrow L_P^0(\Omega; \mathbb{R}), \quad L(t)u^* = \langle Y(t), u^* \rangle$$

defines a cylindrical Lévy process in U . If (b, R, ν) is the characteristics of Y , then the cylindrical characteristics (a, Q, μ) of L is given by

$$a(u^*) = \langle b, u^* \rangle + \int_U \langle u, u^* \rangle (\mathbb{1}_{B_{\mathbb{R}}}(\langle u, u^* \rangle) - \mathbb{1}_{B_U}(u)) \nu(du), \quad Q = R, \quad \mu = \nu.$$

The existence of the integral is derived in Lemma 5.7.

The asymmetry of the classical characteristics and the cylindrical characteristics of Y is due to the fact, that in the cylindrical perspective the entry μ is only a cylindrical measure and therefore, the truncation function $u \mapsto \mathbb{1}_{B_U}(u)$ cannot be integrated with respect to μ . A more illustrative reason is that a classical Lévy process obviously has jumps in the underlying Banach space U , whereas it is not clear in which space the jumps of a cylindrical Lévy process occur.

An appealing way to construct a cylindrical Lévy process is by a series of real valued Lévy processes. We denote here by $\ell^p(\mathbb{R})$ for $p \in [1, \infty]$ the spaces of real valued sequences.

Lemma 4.2. *Let U be a Hilbert space with an orthonormal basis $(e_k)_{k \in \mathbb{N}}$ and let $(\ell_k)_{k \in \mathbb{N}}$ be a sequence of independent, real valued Lévy processes with characteristics (b_k, r_k, ν_k) for $k \in \mathbb{N}$. Then the following are equivalent:*

(a) *For each $(\alpha_k)_{k \in \mathbb{N}} \in \ell^2(\mathbb{R})$ we have*

$$\begin{aligned} \text{(i)} \quad & \sum_{k=1}^{\infty} \mathbb{1}_{B_{\mathbb{R}}}(\alpha_k) |\alpha_k| \left| b_k + \int_{1 < |\beta| \leq |\alpha_k|^{-1}} \beta \nu_k(d\beta) \right| < \infty; \\ \text{(ii)} \quad & (r_k)_{k \in \mathbb{N}} \in \ell^\infty(\mathbb{R}); \\ \text{(iii)} \quad & \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(|\alpha_k \beta|^2 \wedge 1 \right) \nu_k(d\beta) < \infty. \end{aligned}$$

(b) *For each $t \geq 0$ and $u^* \in U^*$ the sum*

$$L(t)u^* := \sum_{k=1}^{\infty} \langle e_k, u^* \rangle \ell_k(t) \quad (4.1)$$

converges P -a.s.

If in this case the set $\{\varphi_{\ell_k(1)} : k \in \mathbb{N}\}$ is equicontinuous at 0, then $(L(t) : t \geq 0)$ defines a cylindrical Lévy process in U with cylindrical characteristics (a, Q, μ) obeying

$$\begin{aligned} a(u^*) &= \sum_{k=1}^{\infty} \langle e_k, u^* \rangle \left(b_k + \int_{\mathbb{R}} \beta \left(\mathbb{1}_{B_{\mathbb{R}}}(\langle e_k, u^* \rangle \beta) - \mathbb{1}_{B_{\mathbb{R}}}(\beta) \right) \nu_k(d\beta) \right), \\ Qu^* &= \sum_{k=1}^{\infty} \langle e_k, u^* \rangle r_k e_k, \quad (\mu \circ (u^*)^{-1})(d\beta) = \sum_{k=1}^{\infty} (\nu_k \circ m_k(u^*)^{-1})(d\beta), \end{aligned}$$

for each $u^ \in U^*$, where $m_k(u^*) : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $m_k(u^*)(\beta) = \langle e_k, u^* \rangle \beta$.*

Proof. Define for an arbitrary sequence $(\alpha_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}$ and $n \in \mathbb{N}$ the partial sum

$$S_n(t) := \sum_{k=1}^n \alpha_k \ell_k(t) \quad \text{for all } t \geq 0. \quad (4.2)$$

It follows by using [33, Pro.11.10], that $(S_n(t) : t \geq 0)$ is a Lévy process with characteristics

$$b^{(n)} := \sum_{k=1}^n b'_k, \quad r^{(n)} := \sum_{k=1}^n \alpha_k^2 r_k, \quad \nu^{(n)}(d\beta) := \sum_{k=1}^n (\nu_k \circ m_{\alpha_k}^{-1})(d\beta),$$

where $m_{\alpha_k} : \mathbb{R} \rightarrow \mathbb{R}$, $m_{\alpha_k}(\beta) = \alpha_k \beta$ and the reals b'_k are defined by

$$b'_k := \alpha_k b_k + \int_{\mathbb{R}} \alpha_k \beta \left(\mathbb{1}_{B_{\mathbb{R}}}(\alpha_k \beta) - \mathbb{1}_{B_{\mathbb{R}}}(\beta) \right) \nu_k(d\beta).$$

For establishing the implication (a) \Rightarrow (b) fix $u^* \in U^*$ and set $\alpha_k := \langle e_k, u^* \rangle$. Due to Condition (i) there exists $b \in \mathbb{R}$ such that $\lim_{n \rightarrow \infty} b^{(n)} = b$. Conditions (ii) and (iii)

guarantee that for every continuous and bounded function $f: \mathbb{R} \rightarrow \mathbb{R}$ we have

$$\begin{aligned} & \int_{\mathbb{R}} f(\beta) \left(r^{(n)} \delta_0(d\beta) + (|\beta|^2 \wedge 1) \nu^{(n)}(d\beta) \right) \\ &= f(0) \sum_{k=1}^n \alpha_k^2 r_k + \sum_{k=1}^n \int_{\mathbb{R}} f(\alpha_k \beta) (|\alpha_k \beta|^2 \wedge 1) \nu_k(d\beta) \\ &\rightarrow f(0) \sum_{k=1}^{\infty} \alpha_k^2 r_k + \sum_{k=1}^{\infty} \int_{\mathbb{R}} f(\alpha_k \beta) (|\alpha_k \beta|^2 \wedge 1) \nu_k(d\beta) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Theorem VII.2.9 and Remark VII.2.10 in [15] imply that the sum $S_n(t)$ converges weakly and therefore P -a.s. to an infinitely divisible random variable $L(t)u^*$ for each $t \geq 0$.

Conversely, it follows from (b) that the sum $S_n(1)$, defined in (4.2), converges weakly for every $(\alpha_k)_{k \in \mathbb{N}} \in \ell^2(\mathbb{R})$. Theorem VII.2.9 in [15] implies that $b^{(n)}$ converges as $n \rightarrow \infty$, i.e.

$$\left| \sum_{k=1}^{\infty} \alpha_k \left(b_k + \int_{\mathbb{R}} \beta (\mathbb{1}_{B_{\mathbb{R}}}(\alpha_k \beta) - \mathbb{1}_{B_{\mathbb{R}}}(\beta)) \nu_k(d\beta) \right) \right| < \infty$$

for every $(\alpha_k)_{k \in \mathbb{N}} \in \ell^2(\mathbb{R})$. Since for each $k \in \mathbb{N}$ the term in the bracket does not depend on the sign of α_k we can choose α_k such that each summand is positive and we obtain

$$\sum_{k=1}^{\infty} |\alpha_k| \left| b_k + \int_{\mathbb{R}} \beta (\mathbb{1}_{B_{\mathbb{R}}}(\alpha_k \beta) - \mathbb{1}_{B_{\mathbb{R}}}(\beta)) \nu_k(d\beta) \right| < \infty,$$

which yields Condition (i) as $|\alpha_k| \leq 1$ for sufficiently large k . Remark VII.2.10 in [15] implies that for every continuous, bounded function $f: \mathbb{R} \rightarrow \mathbb{R}$ the sum

$$\sum_{k=1}^n \left(\alpha_k^2 r_k f(0) + \int_{\mathbb{R}} f(\alpha_k \beta) (|\alpha_k \beta|^2 \wedge 1) \nu_k(d\beta) \right)$$

converges as $n \rightarrow \infty$. Since we can assume that $r_k \geq 0$ for every $k \in \mathbb{N}$, Conditions (ii) and (iii) are implied by choosing $f(\cdot) = 1$, which completes the proof of the equivalence (a) \Leftrightarrow (b).

Clearly, $L(t): U^* \rightarrow L_P^0(\Omega; \mathbb{R})$ is linear. If a sequence $(u_n^*)_{n \in \mathbb{N}} \subseteq U^*$ converges to 0 then $\langle e_k, u_n^* \rangle \rightarrow 0$ as $n \rightarrow \infty$ uniformly in $k \in \mathbb{N}$. The equicontinuity of $\{\varphi_{\ell_k(1)} : k \in \mathbb{N}\}$ implies for each $t \geq 0$ that $\varphi_{\ell_k(t)}(\langle e_k, u_n^* \rangle) \rightarrow 1$ for $n \rightarrow \infty$ uniformly in $k \in \mathbb{N}$. Thus, we obtain

$$\lim_{n \rightarrow \infty} \varphi_{L(t)}(u_n^*) = \lim_{n \rightarrow \infty} \prod_{k=1}^{\infty} \varphi_{\ell_k(t)}(\langle e_k, u_n^* \rangle) = \prod_{k=1}^{\infty} \lim_{n \rightarrow \infty} \varphi_{\ell_k(t)}(\langle e_k, u_n^* \rangle) = 1,$$

which shows the continuity of $L(t)$. Finally, L has weakly independent increments, that is the random variables

$$(L(t_1) - L(t_0))u_1^*, \dots, (L(t_n) - L(t_{n-1}))u_n^*$$

are independent for every $0 \leq t_0 \leq \dots \leq t_n$, $u_1^*, \dots, u_n^* \in U^*$ and $n \in \mathbb{N}$. Since $(L(t)u^* : t \geq 0)$ is a real valued Lévy process for each $u^* \in U^*$ it follows from [1, Le.3.8] that L is a cylindrical Lévy process in U . \square

The convergence (4.1) is called the *weakly P-a.s. convergence* of the sum $\sum e_k \ell_k(t)$ for each $t \geq 0$. If there exists a random variable $Y(t) : \Omega \rightarrow U$ for each $t \geq 0$ such that

$$Y(t) = \sum_{k=1}^{\infty} e_k \ell_k(t) \quad P\text{-a.s. in } U,$$

then the sum is called *strongly P-a.s. convergent*. Obviously, in this case, we have $L(t)u^* = \langle Y(t), u^* \rangle$ for every $u^* \in U^*$ and $t \geq 0$, and $(Y(t) : t \geq 0)$ is a U -valued Lévy process. One can easily show a similar result to Lemma 4.2 but we skip this; a special case can be found in [26, Th.4.13]

Example 4.3. Let ℓ_k be defined by $\ell_k(\cdot) := \sigma_k w_k(\cdot)$ where $(\sigma_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}$ and $(w_k)_{k \in \mathbb{N}}$ is a sequence of independent, real valued standard Brownian motions. Then the series in (4.1) defines a cylindrical Lévy process L if and only if $(\sigma_k)_{k \in \mathbb{N}} \in \ell^\infty$. In this case L is called a *cylindrical Wiener process*. Its covariance operator Q is given by

$$Q : U^* \rightarrow U, \quad Qu^* = \sum_{k=1}^{\infty} \sigma_k^2 \langle e_k, u^* \rangle e_k.$$

This definition of a cylindrical Wiener process is consistent with other definitions which can be found in the literature, see [29].

Example 4.4. For a sequence $(h_k)_{k \in \mathbb{N}}$ of independent, real valued Poisson processes with intensity 1 and a sequence $\sigma := (\sigma_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}$ we define $\ell_k(\cdot) := \sigma_k h_k(\cdot)$. In this case, the sum (4.1) defines a cylindrical Lévy process if and only if $\sigma \in \ell^2$. The sum converges strongly if and only if $\sigma \in \ell^1$. If $(h_k)_{k \in \mathbb{N}}$ is a sequence of independent, real valued compensated Poisson processes with intensity 1, then the sum converges weakly if and only if $\sigma \in \ell^\infty$ and strongly if and only if $\sigma \in \ell^2$.

Example 4.5. Let $(h_k)_{k \in \mathbb{N}}$ be a family of independent, identically distributed, real valued, standardised, symmetric, α -stable Lévy processes h_k for $\alpha \in (0, 2)$. Then the characteristics of h_k is given by $(0, 0, \rho)$ with Lévy measure $\rho(d\beta) = \frac{1}{2} |\beta|^{-1-\alpha} d\beta$. For a sequence $\sigma := (\sigma_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}$ define for each $k \in \mathbb{N}$ the Lévy process $\ell_k(\cdot) := \sigma_k h_k(\cdot)$. Then the characteristics of ℓ_k is given by $(0, 0, \nu_k)$ with

$$\nu_k := \rho \circ m_{\sigma_k}^{-1}, \quad (4.3)$$

where $m_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $m_\alpha \beta = \alpha \beta$ for some $\alpha \in \mathbb{R}$. With this choice of $(\ell_k)_{k \in \mathbb{N}}$ it follows by Lemma 4.2 that the sum (4.1) defines a cylindrical Lévy process if and only if

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(|\alpha_k \beta|^2 \wedge 1 \right) \nu_k(d\beta) = \frac{2}{\alpha(2-\alpha)} \sum_{k=1}^{\infty} |\alpha_k \sigma_k|^\alpha < \infty$$

for every $(\alpha_k)_{k \in \mathbb{N}} \in \ell^2$, which is equivalent to $\sigma \in \ell^{(2\alpha)/(2-\alpha)}$. The cylindrical Lévy process is U -valued, i.e. the sum converges strongly, if and only if $\sigma \in \ell^\alpha$.

Remark 4.6. The publication [28] treats the cylindrical Lévy process introduced in Example 4.5 and it is called *cylindrical stable noise*. This specific example of a cylindrical Lévy process appears also in the publications [6], [18] and [27]. However, since the authors do not follow the cylindrical approach, they do not require that the sum (4.1) is finite, i.e. they do not impose any conditions on the sequence σ . Although this is more general, it does not

match the usual framework, if one understands cylindrical Lévy processes as a generalisation of cylindrical Wiener processes. All the different definitions of cylindrical Wiener processes, one can find in the literature, have in common that the corresponding sum of the form (4.1) converges weakly as in Example 4.3.

Example 4.7. In [8], the authors construct a noise by subordination which they call *Lévy white noise*. In the following result we define this noise in our setting and derive its cylindrical characteristics. In contrast to the original source, our cylindrical approach enables us to introduce this noise without referring to any other space than the underlying Banach space U , which we consider to be more natural.

Lemma 4.8. *If W is a cylindrical Wiener process in U with covariance operator C and ℓ is an independent, real valued subordinator with characteristics $(\alpha, 0, \rho)$ then*

$$L(t)u^* := W(\ell(t))u^* \quad \text{for all } u^* \in U^*, t \geq 0, \quad (4.4)$$

defines a cylindrical Lévy process $(L(t) : t \geq 0)$ with characteristics $(0, Q, \mu)$ given by

$$Q = \alpha C, \quad \mu = (\gamma \otimes \rho) \circ \kappa^{-1},$$

where γ is the canonical Gaussian cylindrical measure on the reproducing kernel Hilbert space H_C of C with embedding $i_C : H_C \rightarrow U$ and

$$\kappa : H_C \times \mathbb{R}_+ \rightarrow U, \quad \kappa(h, s) := \sqrt{s} i_C h.$$

Proof. The very definition of L implies by Lemma 3.8 in [1] using independence of W and ℓ that L is a cylindrical Lévy process. The characteristic function of the subordinator ℓ can be analytically continued, such that for each $t \geq 0$ we obtain the Laplace transform of $\ell(t)$ by

$$E[\exp(-\beta \ell(t))] = \exp(-t\tau(\beta)) \quad \text{for all } \beta > 0,$$

where the Laplace exponent τ is defined by

$$\tau(\beta) := \alpha\beta + \int_0^\infty (1 - e^{-\beta s}) \rho(ds) \quad \text{for all } \beta > 0,$$

see [33, Th.24.11]. Independence of W and ℓ implies that for each $t \geq 0$ and $u^* \in U^*$ the characteristic function $\varphi_{L(t)}$ of $L(t)$ is given by

$$\begin{aligned} \varphi_{L(t)}(u^*) &= \int_0^\infty E[e^{iW(s)u^*}] P_{\ell(t)}(ds) \\ &= \int_0^\infty e^{-\frac{1}{2}s\langle Cu^*, u^* \rangle} P_{\ell(t)}(ds) = \exp\left(-t\tau\left(\frac{1}{2}\langle Cu^*, u^* \rangle\right)\right). \end{aligned} \quad (4.5)$$

By using $C = i_C i_C^*$, $\gamma(H_C) = 1$ and the symmetry of the canonical Gaussian cylindrical measure γ we obtain

$$\begin{aligned} &\int_0^\infty \left(e^{-\frac{1}{2}s\langle Cu^*, u^* \rangle} - 1 \right) \rho(ds) \\ &= \int_0^\infty \int_{H_C} \left(e^{i\sqrt{s}\langle i_C h, u^* \rangle} - 1 \right) \gamma(dh) \rho(ds) \\ &= \int_0^\infty \int_{H_C} \left(e^{i\sqrt{s}\langle i_C h, u^* \rangle} - 1 - i\sqrt{s}\langle i_C h, u^* \rangle \mathbb{1}_{B_{\mathbb{R}}}(\sqrt{s}\langle i_C h, u^* \rangle) \right) \gamma(dh) \rho(ds) \\ &= \int_U \left(e^{i\langle u, u^* \rangle} - 1 - i\langle u, u^* \rangle \mathbb{1}_{B_{\mathbb{R}}}(\langle u, u^* \rangle) \right) ((\gamma \otimes \rho) \circ \kappa^{-1})(du). \end{aligned} \quad (4.6)$$

Note that $(\gamma \otimes \rho) \circ \kappa^{-1}$ is a cylindrical Lévy measure since for each $u^* \in U^*$ we have

$$\begin{aligned} \int_U \left(\langle u, u^* \rangle^2 \wedge 1 \right) ((\gamma \otimes \rho) \circ \kappa^{-1})(du) &= \int_0^\infty \int_{H_C} \left(s \langle i_C h, u^* \rangle^2 \wedge 1 \right) \gamma(dh) \rho(ds) \\ &\leq \int_0^\infty \left(s \int_{H_C} \langle i_C h, u^* \rangle^2 \gamma(dh) \wedge 1 \right) \rho(ds) \\ &= \int_0^\infty \left(s \langle C u^*, u^* \rangle \wedge 1 \right) \rho(ds) < \infty. \end{aligned}$$

The finiteness of the last integral is shown in [33, Th.21.5]. Applying equality (4.6) to the representation (4.5) yields that the characteristic function of $L(t)$ is of the claimed form. \square

The previous example highlights an important difference between cylindrical Wiener and cylindrical Lévy processes. According to the Karhunen-Loève expansion, each cylindrical Wiener processes can be represented by a sum of independent U -valued random variables, see for example [29, Th.20]. However, such kind of representation cannot be expected for the noise constructed in Lemma 4.8.

Example 4.9. Another example of a cylindrical Lévy process is the *impulsive cylindrical process* on $L_\lambda^2(\mathcal{O}; \mathbb{R})$, which is introduced in the monograph [26]. In our work [1] we show that also this kind of a noise can be understood as a specific example of a cylindrical Lévy approach in our general approach.

5 Stochastic integration

In this section, U and V are separable Banach spaces and $(L(t) : t \geq 0)$ denotes a cylindrical Lévy process in U with characteristics (a, Q, μ) . Let $f : [0, T] \rightarrow \mathcal{L}(U, V)$ be a deterministic function, where $\mathcal{L}(U, V)$ denotes the space of linear, bounded functions from U to V . The aim of this section is to define a stochastic integral

$$I_A := \int_A f(s) dL(s)$$

as a V -valued random variable for each Borel set $A \subseteq [0, T]$.

Our approach is based on the idea that the random variable I_A , if it exists, must be infinitely divisible and thus, its probability distribution is uniquely described by its characteristics, say (b_A, R_A, ν_A) . If we have a candidate for the characteristics of I_A and if there are conditions known (such as in Hilbert spaces) guaranteeing the existence of an infinitely divisible random variable in terms of its prospective characteristics, then we can describe the class of integrable functions $f : [0, T] \rightarrow \mathcal{L}(U, V)$. This approach works in every separable Banach space V in which explicit conditions on the characteristics are known guaranteeing the existence of a corresponding infinitely divisible measure.

In order to have a candidate for the characteristics of I_A at hand for formulating the conditions on integrability, we first introduce a cylindrical random variable $Z_A : V^* \rightarrow L_P^0(\Omega; \mathbb{R})$ as a *cylindrical integral* of f . Then we call f *integrable with respect to L* if for each Borel set $A \subseteq [0, T]$ the cylindrical integral Z_A is induced by a classical random variable $I_A : \Omega \rightarrow V$, i.e.

$$Z_A v^* = \langle I_A, v^* \rangle \quad \text{for all } v^* \in V^*.$$

In this way one can think of I_A as a stochastic Pettis integral.

For a well defined integral we expect that

$$\left\langle \int_A f(s) dL(s), v^* \right\rangle = \int_A f^*(s) v^* dL(s) \quad \text{for all } v^* \in V^*.$$

Thus, in a first step, we introduce a real valued stochastic integral for U^* -valued functions with respect to a cylindrical Lévy process L . For this purpose, we initially consider simple U^* -valued functions. A deterministic function $g: [0, T] \rightarrow U^*$ is called simple if there is a partition $0 = t_0 \leq t_1 \leq \dots \leq t_m = T$ such that g is constant on the open interval (t_k, t_{k+1}) for each $k = 0, \dots, m-1$. The space of all simple functions is denoted by $\mathcal{S}([0, T]; U^*)$ and it is endowed with the supremum norm

$$\|g\|_\infty := \sup_{s \in [0, T]} \|g(s)\|.$$

Let $\mathcal{G}([0, T]; U^*)$ denote the space of deterministic regulated functions; these are all functions $g: [0, T] \rightarrow U^*$ such that for every $t \in (0, T)$ there exists the limit of g on the left and on the right of t and on the right of 0 and on the left of T . In other words, each regulated function has only discontinuities of the first kind. It is shown in [5, Ch.II.1.3] or [11, Ch.VII.6] that a function g is regulated if and only if it can be uniformly approximated by step functions. In particular, regulated functions are bounded and the space $\mathcal{G}([0, T]; U^*)$ can be equipped with the supremum norm, which turns it into a Banach space.

For a simple function $g \in \mathcal{S}([0, T]; U^*)$ which attains the value u_k^* on the interval (t_k, t_{k+1}) for $k = 0, \dots, m-1$ define a mapping $J: \mathcal{S}([0, T]; U^*) \rightarrow L_P^0(\Omega; \mathbb{R})$ by

$$J(g) := \sum_{k=0}^{m-1} (L(t_{k+1}) - L(t_k))(u_k^*). \quad (5.1)$$

In order to show that the mapping J is continuous we need the following result.

Lemma 5.1. *If $\Psi: U^* \rightarrow \mathbb{C}$ is the symbol of a cylindrical Lévy process L then the mapping*

$$\mathcal{G}([0, T]; U^*) \rightarrow L^1([0, T]; \mathbb{C}), \quad g \mapsto \Psi(g(\cdot))$$

is continuous.

Proof. Continuity of Ψ and Lemma 3.2 guarantee that $\Psi(g(\cdot)) \in L^1([0, T]; \mathbb{C})$ for $g \in \mathcal{G}([0, T]; U^*)$. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{G}([0, T]; U^*)$ converging to $g \in \mathcal{G}([0, T]; U^*)$. Recall that (a, Q, μ) denotes the cylindrical characteristics of L and Ψ is given in (3.1). Inequality (3.11) shows that $a: U^* \rightarrow \mathbb{R}$ maps bounded sets into bounded sets and thus, Lebesgue's dominated convergence theorem implies

$$\lim_{n \rightarrow \infty} \int_0^T |a(g_n(s)) - a(g(s))| ds = 0.$$

Another application of Lebesgue's dominated convergence theorem shows

$$\lim_{n \rightarrow \infty} \int_0^T |\langle g_n(s), Qg_n(s) \rangle - \langle g(s), Qg(s) \rangle| ds = 0.$$

Define for each $n \in \mathbb{N}$ the function

$$h_n: [0, T] \rightarrow \mathbb{C}, \quad h_n(s) = \int_U \left(e^{i\langle u, g_n(s) \rangle} - 1 - i\langle u, g_n(s) \rangle \mathbb{1}_{B_{\mathbb{R}}}(\langle u, g_n(s) \rangle) \right) \mu(du),$$

and the function

$$f: \mathbb{R} \rightarrow \mathbb{C}, \quad f(\beta) = \begin{cases} \frac{e^{i\beta} - 1 - i\beta \mathbf{1}_{B_{\mathbb{R}}}(\beta)}{\beta^2 \wedge 1}, & \text{if } \beta \neq 0, \\ -\frac{1}{2}, & \text{if } \beta = 0. \end{cases}$$

Clearly, the function f is bounded and continuous. Lemma 3.1 guarantees

$$\begin{aligned} \sup_{n \in \mathbb{N}} \sup_{s \in [0, T]} |h_n(s)| &= \sup_{n \in \mathbb{N}} \sup_{s \in [0, T]} \left| \int_U f(\langle u, g_n(s) \rangle) (\langle u, g_n(s) \rangle^2 \wedge 1) \mu(du) \right| \\ &\leq \|f\|_{\infty} \sup_{n \in \mathbb{N}} \sup_{s \in [0, T]} \int_U (\langle u, g_n(s) \rangle^2 \wedge 1) \mu(du) \\ &< \infty. \end{aligned}$$

Since $g_n(s) \rightarrow g(s)$ for each $s \in [0, T]$, it follows from (3.5):

$$\begin{aligned} \lim_{n \rightarrow \infty} h_n(s) &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(\beta) (|\beta|^2 \wedge 1) (\mu \circ g_n(s)^{-1})(d\beta) \\ &= \int_{\mathbb{R}} f(\beta) (|\beta|^2 \wedge 1) (\mu \circ g(s)^{-1})(d\beta) \\ &=: h(s). \end{aligned}$$

Lebesgue's dominated convergence theorem implies

$$\lim_{n \rightarrow \infty} \int_0^T |h_n(s) - h(s)| ds = 0,$$

which completes the proof. \square

Lemma 5.2. *The operator $J: \mathcal{S}([0, T]; U^*) \rightarrow L_P^0(\Omega; \mathbb{R})$ defined in (5.1) is continuous where $L_P^0(\Omega; \mathbb{R})$ is equipped with the topology of convergence in probability.*

Proof. Let $(g_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}([0, T]; U^*)$ be a sequence converging to g in $\mathcal{S}([0, T]; U^*)$. Then, by linearity of J it follows that $J(g_n)$ converges to $J(g)$ in probability if and only if $J(g_n - g) \rightarrow 0$ in probability. However, the latter convergence occurs if and only if $J(g_n - g) \rightarrow 0$ weakly.

Independent increments of L yields that the characteristic function of $J(g_n)$ is given by

$$\varphi_{J(g_n)}: \mathbb{R} \rightarrow \mathbb{C}, \quad \varphi_{J(g_n)}(\beta) = \exp \left(\int_0^T \Psi(\beta g_n(s)) ds \right).$$

Consequently, it follows from Lemma 5.1 that $\varphi_{J(g_n)}(\beta)$ converges to $\varphi_{J(g)}(\beta)$ for all $\beta \in \mathbb{R}$, which completes the proof. \square

The mapping $J: \mathcal{S}([0, T]; U^*) \rightarrow L_P^0(\Omega; \mathbb{R})$ is linear and uniformly continuous, since the metric of $L_P^0(\Omega; \mathbb{R})$ is translation invariant. The principle of extension by continuity ([12, Th.I.6.17]) enables us to extend the mapping J to the space $\mathcal{G}([0, T]; U^*)$, i.e. we define

$$J(g) := \lim_{n \rightarrow \infty} J(g_n) \quad \text{in } L_P^0(\Omega; \mathbb{R}),$$

where $(g_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}([0, T]; U^*)$ is chosen such that $g_n \rightarrow g$ in $\mathcal{G}([0, T]; U^*)$. Lemma 5.1 implies that the characteristic function of $J(g)$ is given by

$$\varphi_{J(g)}: \mathbb{R} \rightarrow \mathbb{C}, \quad \varphi_{J(g)}(\beta) = \exp \left(\int_0^T \Psi(\beta g(s)) ds \right). \quad (5.2)$$

Now we come back to the original aim to introduce a stochastic integral for integrands with values in $\mathcal{L}(U, V)$ as a genuine V -valued random variable. For that purpose, let $f: [0, T] \rightarrow \mathcal{L}(U, V)$ be a function and for each $s \in [0, T]$ denote by $f^*(s)$ the adjoint of $f(s): U \rightarrow V$. We say that the function $f: [0, T] \rightarrow \mathcal{L}(U, V)$ is *weakly in* $\mathcal{G}([0, T]; U^*)$ if $f^*(\cdot)v^*$ is in $\mathcal{G}([0, T]; U^*)$ for each $v^* \in V^*$. Clearly, the function $s \mapsto f^*(s)v^*$ is measurable for each $v^* \in V^*$.

Lemma 5.3. *If $\tau: [0, T] \rightarrow \mathbb{R}$ is measurable and bounded and $g \in \mathcal{G}([0, T]; U^*)$ then $\tau(\cdot)g(\cdot) \in \mathcal{G}([0, T]; U^*)$.*

Proof. Since τ is bounded there exist simple functions τ_n converging uniformly to τ . If $(g_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}([0, T]; U^*)$ converges uniformly to g it follows

$$\|\tau g - \tau_n g_n\|_\infty \leq \|\tau\|_\infty \|g - g_n\|_\infty + \|\tau - \tau_n\|_\infty \|g_n\|_\infty \rightarrow 0.$$

Thus, τg can be uniformly approximated by simple functions, which completes the proof. \square

Lemma 5.3 guarantees that if $A \in \mathcal{B}([0, T])$ and $g \in \mathcal{G}([0, T]; U^*)$ then $\mathbb{1}_A(\cdot)g(\cdot) \in \mathcal{G}([0, T]; U^*)$.

Lemma 5.4. *If $f: [0, T] \rightarrow \mathcal{L}(U, V)$ is weakly in $\mathcal{G}([0, T]; U^*)$ then for each $A \in \mathcal{B}([0, T])$*

$$Z_A: V^* \rightarrow L_P^0(\Omega; \mathbb{R}), \quad Z_A v^* := J(\mathbb{1}_A(\cdot)f^*(\cdot)v^*)$$

defines an infinitely divisible cylindrical random variable with characteristic function

$$\varphi_{Z_A}: V^* \rightarrow \mathbb{C}, \quad \varphi_{Z_A}(v^*) = \exp\left(\int_A \Psi(f^*(s)v^*) ds\right).$$

Furthermore, the cylindrical characteristics (b_A, r_A, ν_A) of Z_A is given by

$$b_A: V^* \rightarrow \mathbb{R}, \quad b_A(v^*) := \int_A a(f^*(s)v^*) ds, \quad (5.3)$$

$$r_A: V^* \rightarrow \mathbb{R}, \quad r_A(v^*) = \int_A \langle f^*(s)v^*, Q f^*(s)v^* \rangle ds, \quad (5.4)$$

$$\nu_A: \mathcal{Z}(V) \rightarrow [0, \infty], \quad \nu_A = (\mu \otimes \text{leb}) \circ \chi_A^{-1}, \quad (5.5)$$

where $\chi_A: [0, T] \times U \rightarrow V$ is defined by $\chi_A(s, u) := \mathbb{1}_A(s)f(s)u$.

Proof. Since $\|f^*(\cdot)v^*\|_\infty < \infty$ for all $v^* \in V^*$, the uniform boundedness principle implies

$$m := \sup_{s \in [0, T]} \|f^*(s)\|_{V^* \rightarrow U^*} < \infty. \quad (5.6)$$

Consequently, we obtain $\|\mathbb{1}_A(\cdot)f^*(\cdot)v^*\|_\infty \leq m \|v^*\|$ which implies the continuity of $Z_A: V^* \rightarrow L_P^0(\Omega; \mathbb{R})$ by the continuity of $J: \mathcal{G}([0, T]; U^*) \rightarrow L_P^0(\Omega; \mathbb{R})$. The form of the characteristic function follows immediately from (5.2). \square

The cylindrical random variable Z_A is called the *cylindrical integral of f on A* . However, we want to define a genuine V -valued random variable as the stochastic integral of f which we achieve in the following way:

Definition 5.5. A function $f: [0, T] \rightarrow \mathcal{L}(U, V)$ is called *stochastically integrable w.r.t. L* if f is weakly in $\mathcal{G}([0, T]; U^*)$ and if for each $A \in \mathcal{B}([0, T])$ there exists an $I_A \in L_P^0(\Omega; V)$ such that

$$\langle I_A, v^* \rangle = Z_A v^* \quad \text{for all } v^* \in V^*, \quad (5.7)$$

where Z_A denotes the cylindrical integral of f on A . In this case I_A is denoted by

$$\int_A f(s) dL(s) := I_A.$$

The existence of a random variable $I_A \in L_P^0(\Omega; V)$ satisfying (5.7) is called that Z_A is *induced by I_A* . Such a random variable I_A exists if and only if the cylindrical distribution of Z_A extends to a measure, see [35, Th.IV.2.5]. Our approach by the cylindrical integral enables us to give sufficient and necessary conditions for the extension of the cylindrical distribution of Z_A to a measure. In an arbitrary Banach space, as in the next Theorem, these conditions are rather abstract. However, in more specific spaces, such as Hilbert spaces or Banach spaces of type $p \in [1, 2]$, some or even all of these conditions can be simplified significantly. We demonstrate this by a few subsequent corollaries.

Theorem 5.6. *Let $f: [0, T] \rightarrow \mathcal{L}(U, V)$ be a function which is weakly in $\mathcal{G}([0, T]; U^*)$. Then f is stochastically integrable if and only if the following is satisfied:*

- (1) *the mapping T_a is weak*-weakly sequentially continuous where*

$$T_a: V^* \rightarrow L^1([0, T]; \mathbb{R}), \quad T_a v^* = a(f^*(\cdot)v^*); \quad (5.8)$$

- (2) *there exists a Gaussian covariance operator $R: V^* \rightarrow V$ satisfying*

$$\langle v^*, Rv^* \rangle = \int_0^T \langle f^*(s)v^*, Qf^*(s)v^* \rangle ds \quad \text{for all } v^* \in V^*; \quad (5.9)$$

- (3) *for $\chi: [0, T] \times U \rightarrow V$ defined by $\chi(s, u) := f(s)u$, the cylindrical measure*

$$\nu: \mathcal{Z}(V) \rightarrow [0, \infty], \quad \nu = (\mu \otimes \text{leb}) \circ \chi^{-1}, \quad (5.10)$$

extends to a measure and Lévy measure on $\mathcal{B}(V)$.

Note that although stochastic integrability of f requires that the cylindrical integral Z_A is induced by a random variable I_A for all $A \in \mathcal{B}([0, T])$, Conditions (5.9) and (5.10) can be considered as conditions only for $A = [0, T]$. Condition (5.8) is the best sufficient and necessary prescription for the first term of the characteristics available, in order to guarantee integrability for the following reason: if L is a genuine Lévy process with classical characteristics $(b, 0, 0)$ for some $b \in U$, then stochastic integrability of the function f according to Definition 5.5 reduces to Pettis integrability of the function $f(\cdot)b$. In this case, $a(\cdot) = \langle b, \cdot \rangle$ and Condition (5.8) is known to be equivalent to Pettis integrability of $f(\cdot)b$, see [24, Th.4.1]. In Lemma 5.8 we explain this equivalence more general for genuine Lévy processes with arbitrary characteristics (b, Q, μ) .

In light of Lemma 5.4, Conditions (5.9) and (5.10) seem to be a straightforward conclusion, but the change from a cylindrical to a genuine infinitely divisible measure requires some further arguments. The correction term between the classical Lévy-Khintchine formula in (2.1) and the cylindrical version in (3.1) is considered in the following lemma.

Lemma 5.7. *If ξ is a Lévy measure on $\mathcal{B}(V)$ then the function*

$$\Delta_\xi: V^* \rightarrow \mathbb{R}, \quad \Delta_\xi(v^*) := \int_V \langle v, v^* \rangle (\mathbb{1}_{B_V}(v) - \mathbb{1}_{B_{\mathbb{R}}}(\langle v, v^* \rangle)) \xi(dv),$$

is well defined and satisfies $\Delta_\xi(v_n^) \rightarrow 0$ for a sequence $(v_n^*)_{n \in \mathbb{N}} \subseteq V^*$ converging weakly* to 0.*

Proof. Let $v^* \in V^*$ and define $D(v^*) := \{v \in V : |\langle v, v^* \rangle| \leq 1\}$. It follows for each $v \in V$ that

$$\begin{aligned} |\langle v, v^* \rangle| |\mathbb{1}_{B_V}(v) - \mathbb{1}_{B_{\mathbb{R}}}(\langle v, v^* \rangle)| &= |\langle v, v^* \rangle| \left(\mathbb{1}_{B_V \cap (D(v^*))^c}(v) + \mathbb{1}_{B_V^c \cap D(v^*)}(v) \right) \\ &\leq |\langle v, v^* \rangle|^2 \mathbb{1}_{B_V}(v) + \mathbb{1}_{B_V^c}(v). \end{aligned} \quad (5.11)$$

Proposition 5.4.5 in [17] guarantees

$$\int_V |\langle v, v^* \rangle| |\mathbb{1}_{B_V}(v) - \mathbb{1}_{B_{\mathbb{R}}}(\langle v, v^* \rangle)| \xi(dv) \leq \int_{B_V} \langle v, v^* \rangle^2 \xi(dv) + \xi(B_V^c) < \infty, \quad (5.12)$$

which shows that the function Δ_ξ is well defined. Let $(v_n^*)_{n \in \mathbb{N}} \subseteq V^*$ be a sequence converging weakly* to 0. An analogue estimate as in (5.11) shows for all $n \in \mathbb{N}$:

$$\begin{aligned} \int_V |\langle v, v_n^* \rangle| |\mathbb{1}_{B_V}(v) - \mathbb{1}_{B_{\mathbb{R}}}(\langle v, v_n^* \rangle)| \xi(dv) \\ \leq \int_{B_V} |\langle v, v_n^* \rangle|^2 \xi(dv) + \int_{B_V^c} \mathbb{1}_{D(v_n^*)}(v) |\langle v, v_n^* \rangle| \xi(dv). \end{aligned} \quad (5.13)$$

For $\alpha := \sup_{n \in \mathbb{N}} \|v_n^*\|$ define the mapping $m_\alpha: V \rightarrow V$ by $m_\alpha(v) := \alpha^{-1}v$. Then $\tilde{\xi}_\alpha := (\xi + \xi^-) \circ m_\alpha^{-1}$ is a Lévy measure on $\mathcal{B}(V)$, and thus there exists an infinitely divisible measure ϑ on $\mathcal{B}(V)$ with characteristics $(0, 0, \tilde{\xi}_\alpha)$. The inequality $1 - \cos(\beta) \geq \frac{1}{3}\beta^2$ for all $|\beta| \leq 1$ implies by using the symmetry of $\tilde{\xi}_\alpha$ that the characteristic function of ϑ satisfies for each $n \in \mathbb{N}$:

$$\begin{aligned} \varphi_\vartheta(v_n^*) &= \exp \left(- \int_V (1 - \cos(\langle v, v_n^* \rangle)) (\xi + \xi^-) \circ m_\alpha^{-1}(dv) \right) \\ &\leq \exp \left(- \int_{B_V} (1 - \cos(\alpha^{-1} \langle v, v_n^* \rangle)) (\xi + \xi^-)(dv) \right) \\ &\leq \exp \left(- \frac{2}{3\alpha^2} \int_{B_V} |\langle v, v_n^* \rangle|^2 \xi(dv) \right). \end{aligned}$$

Since the characteristic function φ_ϑ is weakly* sequentially continuous we obtain

$$\int_{B_V} |\langle v, v_n^* \rangle|^2 \xi(dv) \leq -\frac{3\alpha^2}{2} \ln(\varphi_\vartheta(v_n^*)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since $\xi(B_V^c) < \infty$ and $\mathbb{1}_{D(v_n^*)}(v) |\langle v, v_n^* \rangle| \leq 1$ for all $n \in \mathbb{N}$ and $v \in B_V^c$, Lebesgue's theorem of dominated convergence implies

$$\int_{B_V^c} \mathbb{1}_{D(v_n^*)}(v) |\langle v, v_n^* \rangle| \xi(dv) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which completes the proof by (5.13). \square

Lemma 5.8. *Let $f: [0, T] \rightarrow \mathcal{L}(U, V)$ be a function which is weakly in $\mathcal{G}([0, T]; U^*)$. Then Condition (5.8) is satisfied if and only if*

- (1) *for every sequence $(v_n^*)_{n \in \mathbb{N}} \subseteq V^*$ converging weakly* to 0 and $A \in \mathcal{B}([0, T])$ we have*

$$\lim_{n \rightarrow \infty} \int_A a(f^*(s)v_n^*) ds = 0. \quad (5.14)$$

If L is a genuine Lévy process with characteristics (b, Q, μ) then its cylindrical characteristics (a, Q, μ) satisfies $a = \langle b, \cdot \rangle - \Delta_\mu$ and (1) is equivalent to

- (2) *the mapping $t \mapsto f(t)b$ is Pettis integrable.* (5.15)

Proof. Since $a: U^* \rightarrow \mathbb{R}$ maps bounded sets to bounded sets according to (3.11), Condition (5.14) implies (5.8) by standard arguments.

For the second part assume that L is a genuine Lévy process and we adopt the notation Δ_μ from Lemma 5.7 but for the Lévy measure μ on $\mathcal{B}(U)$. The first entries of the classical characteristics (b, Q, μ) and of the cylindrical characteristics (a, Q, μ) obey $\langle b, u^* \rangle = a(u^*) + \Delta_\mu(u^*)$ for all $u^* \in U^*$, which implies for all $v^* \in V^*$ and $h \in L^\infty([0, T]; \mathbb{R})$ the identity

$$\int_0^T h(s) \langle f(s)b, v^* \rangle ds = \int_0^T h(s) a(f^*(s)v^*) ds + \int_0^T h(s) \Delta_\mu(f^*(s)v^*) ds. \quad (5.16)$$

Let $m := \sup_{s \in [0, T]} \|f^*(s)\|_{V^* \rightarrow U^*}$ be defined as in (5.6) and let $(v_n^*)_{n \in \mathbb{N}} \subseteq V^*$ be a sequence weakly* converging to 0 with setting $\alpha := \sup_{n \in \mathbb{N}} \|v_n^*\|$. From (5.12) and [17, Pro.5.4.5] we conclude

$$\sup_{n \in \mathbb{N}} \sup_{s \in [0, T]} |\Delta_\mu(f^*(s)v_n^*)| \leq \sup_{\|u^*\| \leq \alpha m} \int_{B_U} \langle u, u^* \rangle^2 \mu(du) + \mu(B_U^c) < \infty.$$

By applying Lebesgue's theorem of dominated convergence and Lemma 5.7, we obtain for every $h \in L^\infty([0, T]; \mathbb{R})$ that

$$\left| \int_0^T h(s) \Delta_\mu(f^*(s)v_n^*) ds \right| \leq \|h\|_\infty \int_0^T |\Delta_\mu(f^*(s)v_n^*)| ds \rightarrow 0. \quad (5.17)$$

Let $S_b: V^* \rightarrow L^1([0, T]; \mathbb{R})$ denote the mapping defined by $S_b v^* = \langle f(s)b, v^* \rangle$. It follows from (5.16) and (5.17) that S_b is weak*-weakly sequentially continuous if and only if the mapping T_a , defined in (5.8), is weak*-weakly sequentially continuous. Since the mapping $t \mapsto f(t)b$ is Pettis integrable if and only if S_b is weak*-weakly continuous according to [24, Th.4.1] and since V is separable, the proof is completed. \square

Proof. (Theorem 5.6). Sufficiency: according to (3.4) the cylindrical Lévy process L can be decomposed into $L(t) = W(t) + P(t)$ for all $t \geq 0$, where W and P are independent, cylindrical Lévy processes with characteristics $(0, Q, 0)$ and $(a, 0, \mu)$, respectively. For the cylindrical integral Z_A of f on $A \in \mathcal{B}([0, T])$ we obtain

$$Z_A v^* = Z_A^W v^* + Z_A^P v^* \quad \text{for all } v^* \in V^*,$$

where Z_A^W is the cylindrical integral w.r.t. W on A and Z_A^P w.r.t. P on A .

Lemma 5.4 implies that the cylindrical random variable Z_A^W is Gaussian with covariance r_A defined in (5.4). Since the covariance satisfies

$$r_A(v^*) \leq \int_0^T \langle f^*(s)v^*, Qf^*(s)v^* \rangle ds = \langle v^*, Rv^* \rangle \quad \text{for all } v^* \in V^*,$$

Theorem 3.3 implies that the cylindrical distribution of Z_A^W extends to a measure on $\mathcal{B}(V)$. It follows from Theorem IV.2.5 in [35] that there exists a random variable $I_A^W \in L_P^0(\Omega; V)$ with $\langle I_A^W, v^* \rangle = Z_A^W v^*$ for all $v^* \in V^*$.

According to Lemma 5.4 the cylindrical random variable Z_A^P has the Lévy measure ν_A , which obeys for every set $C \in \mathcal{Z}(V)$ the inequality

$$\nu_A(C) = \int_A \int_U \mathbb{1}_C(f(s)u) \nu(du) ds \leq \int_0^T \int_U \mathbb{1}_C(f(s)u) \nu(du) ds = \nu(C).$$

Theorem 3.4 implies that ν_A extends to a Lévy measure on $\mathcal{B}(V)$, which is also denoted by ν_A . Thus, there exists a probability measure ϑ_A on $\mathcal{B}(V)$ with characteristic function

$$\varphi_{\vartheta_A}(v^*) = \exp \left(\int_V \left(e^{i\langle v, v^* \rangle} - 1 - i\langle v, v^* \rangle \mathbb{1}_{B_V}(v) \right) \nu_A(dv) \right) \quad \text{for all } v^* \in V^*. \quad (5.18)$$

Define the function

$$c_A: V^* \rightarrow \mathbb{R}, \quad c_A(v^*) := \int_A a(f^*(s)v^*) ds + \Delta_{\nu_A}(v^*),$$

where the function Δ_{ν_A} is defined in Lemma 5.7. Let X_A be a V -valued random variable with probability distribution ϑ_A . It follows from Lemma 5.4 and (5.18) that

$$Z_A^P v^* \stackrel{d}{=} c_A(v^*) + \langle X_A, v^* \rangle \quad \text{for all } v^* \in V^*, \quad (5.19)$$

where $\stackrel{d}{=}$ denotes equality in distribution. Linearity and continuity of Z_A^P and $\langle X_A, \cdot \rangle$ implies that $c_A \in V^{**}$. It follows from Condition (5.8), Lemma 5.7 and Lemma 5.8 that c_A is weakly* sequentially continuous. Since V is separable, we obtain that c_A is weakly* continuous ([19, Co.2.7.10]), which implies $c_A \in V$. Consequently, we can define the Dirac measure δ_{c_A} at the point $c_A \in V$. It follows from (5.19) that the cylindrical distribution of Z_A^P extends to the convolution $\delta_{c_A} * \vartheta_A$. Consequently, [35, Th.IV.2.5] guarantees that there exists a random variable $I_A^P \in L_P^0(\Omega; V)$ with $\langle I_A^P, v^* \rangle = Z_A^P v^*$ for all $v^* \in V^*$, which shows the stochastic integrability of f .

Necessity: let $I_A \in L_P^0(\Omega; V)$ denote the stochastic integral of f on A and let (c_A, S_A, ξ_A) be the characteristics of the infinitely divisible random variable I_A . Then, due to the uniqueness of the Lévy-Khintchine formula in \mathbb{R} , for each $v^* \in V^*$ the characteristics of the real valued random variables $\langle I_A, v^* \rangle$ and $Z_A v^*$ coincide which results in

$$\langle c_A, v^* \rangle - \Delta_{\xi_A}(v^*) = \int_A a(f^*(s)v^*) ds, \quad (5.20)$$

$$\langle v^*, S_A v^* \rangle = \int_A \langle f^*(s)v^*, Q f^*(s)v^* \rangle ds, \quad (5.21)$$

$$\xi_A \circ (v^*)^{-1} = \left((\mu \otimes \text{leb}) \circ \chi_A^{-1} \right) \circ (v^*)^{-1}. \quad (5.22)$$

Here, we obtain the characteristics of $\langle I_A, v^* \rangle$ on the left hand side by a standard calculation for the transform of an infinitely divisible measure under a linear mapping (see e.g. [33, Pro.11.10]), whereas the characteristics of $Z_A v^*$ on the right hand side is given in Lemma 5.4. Equation (5.20) shows Condition (5.8) due to Lemma 5.7 and Lemma 5.8. By choosing $A := [0, T]$, identity (5.21) implies Condition (5.9).

By using (5.22) and [33, Pro.11.10], it follows from linearity that the \mathbb{R}^n -valued infinitely divisible random variables $(\langle I_A, v_1^* \rangle, \dots, \langle I_A, v_n^* \rangle)$ and $(Z_A v_1^*, \dots, Z_A v_n^*)$ have the same Lévy measures for all $v_1^*, \dots, v_n^* \in V^*$ and $n \in \mathbb{N}$, i.e.

$$\xi_A \circ \pi_{v_1^*, \dots, v_n^*}^{-1} = \left((\mu \otimes \text{leb}) \circ \chi_A^{-1} \right) \circ \pi_{v_1^*, \dots, v_n^*}^{-1}.$$

Consequently, by choosing $A = [0, T]$ we have $\nu = (\mu \otimes \text{leb}) \circ \chi_A^{-1}$ and the image cylindrical measure ν extends to the Lévy measure ξ_A . \square

Remark 5.9. In the work [32] together with van Gaans, we developed a stochastic integral for deterministic integrands w.r.t. martingale valued measures in Banach spaces, i.e. in particular with respect to the compensated Poisson random measure of a classical Lévy process in a Banach space U . The integrability of a function is described in terms of the appropriate convergence of a random series, which is very similar to the case of γ -radonifying operators. This approach cannot be applied to cylindrical Lévy processes as they do not satisfy an Itô-Lévy decomposition in the underlying Banach space but only in \mathbb{R} , which is then depending on the argument in U^* . Both integrals in the current work and in [32] underly a kind of a stochastic version of the Pettis integral.

As mentioned before, we simplify the conditions in Theorem 5.6 in some more specific spaces. We begin with the most important case of a Hilbert space.

Theorem 5.10. *Assume that V is a Hilbert space with orthonormal basis $(e_k)_{k \in \mathbb{N}}$ and let $f: [0, T] \rightarrow \mathcal{L}(U, V)$ be a function which is weakly in $\mathcal{G}([0, T]; U^*)$. Then f is stochastically integrable if and only if the following is satisfied:*

- (1) *the mapping T_a is weak-weakly sequentially continuous where*

$$T_a: V^* \rightarrow L^1([0, T]; \mathbb{R}), \quad T_a v^* = a(f^*(\cdot) v^*); \quad (5.23)$$

$$(2) \int_0^T \text{tr} [f(s) Q f^*(s)] ds < \infty; \quad (5.24)$$

$$(3) \limsup_{m \rightarrow \infty} \sup_{n \geq m} \int_0^T \int_U \left(\sum_{k=m}^n \langle u, f^*(s) e_k \rangle^2 \wedge 1 \right) \mu(du) ds = 0. \quad (5.25)$$

Proof. The closed graph theorem shows that

$$\langle v^*, R w^* \rangle = \int_0^T \langle f^*(s) v^*, Q f^*(s) w^* \rangle ds \quad \text{for all } v^*, w^* \in V^*,$$

defines a positive, symmetric and bounded operator $R: V^* \rightarrow V$. By applying Tonelli's theorem we can conclude that the operator R is of trace class if and only if Condition (5.24) is satisfied. Since the space of Gaussian covariance operators in Hilbert spaces coincide with the space of trace class operators by [35, Th.IV.2.4], we have established the equivalence of Conditions (5.9) and (5.24).

If f is stochastically integrable then Theorem 5.6 implies that the cylindrical measure ν , defined in (5.10), extends to a measure and it is a Lévy measure in V . Since V is a Hilbert space, the latter implies (see [25, Th.VI.4.10]), that

$$\int_V (\|v\|^2 \wedge 1) \nu(dv) < \infty,$$

which shows Condition (5.25).

It remains to show that (5.25) implies Condition (5.10), for which we can assume that the cylindrical characteristics of L is of the form $(a, 0, \mu)$. We define the space $V_n := \text{span}\{e_1, \dots, e_n\}$ and we denote by $\pi_n: V \rightarrow V$ the orthogonal projection on V_n for each $n \in \mathbb{N}$. Let Z_A denote the cylindrical integral of f on $A \in \mathcal{B}([0, T])$, which has the characteristics $(b_A, 0, \nu_A)$ according to Lemma 5.4. If Z'_A denotes an independent copy of Z_A then $\tilde{Z}_A := Z_A + Z'_A$ is a cylindrical random variable with characteristics $(0, 0, \nu_A + \nu_A^-)$. Since π_n is a Hilbert-Schmidt operator, the cylindrical distribution of $\tilde{Z}_A \circ \pi_n^*$ extends to a probability measure ϑ_n on $\mathcal{B}(V)$ due to [35, Th.VI.5.2], which is infinitely divisible with characteristics $(0, 0, \xi_n)$ where $\xi_n := (\nu_A + \nu_A^-) \circ \pi_n^{-1}$. By using the inequality $1 - \cos \beta \leq 2(\beta^2 \wedge 1)$ for all $\beta \in \mathbb{R}$ we obtain for every $v^* \in V$ that

$$\begin{aligned} 1 - \varphi_{\vartheta_n}(v^*) &= 1 - \exp \left(\int_V (\cos(\langle v, v^* \rangle) - 1) \xi_n(dv) \right) \\ &\leq \int_V (1 - \cos(\langle v, v^* \rangle)) \xi_n(dv) \leq 2 \int_V (\langle v, v^* \rangle^2 \wedge 1) \xi_n(dv). \end{aligned}$$

Let g_m denote the density of the standard normal distribution on $\mathcal{B}(\mathbb{R}^m)$. For every $m, n \in \mathbb{N}$ with $m \leq n$ it follows that

$$\begin{aligned} &\int_{\mathbb{R}^{n-m+1}} (1 - \text{Re } \varphi_{\vartheta_n}(\beta_m e_m + \dots + \beta_n e_n)) g_{n-m+1}(\beta_m, \dots, \beta_n) d\beta_m \dots d\beta_n \\ &\leq 2 \int_{\mathbb{R}^{n-m+1}} \int_V \left(\left| \sum_{k=m}^n \beta_k \langle v, e_k \rangle \right|^2 \wedge 1 \right) \xi_n(dv) g_{n-m+1}(\beta_m, \dots, \beta_n) d\beta_m \dots d\beta_n \\ &\leq 2 \int_V \left(\left(\int_{\mathbb{R}^{n-m+1}} \left[\left| \sum_{k=m}^n \beta_k \langle v, e_k \rangle \right|^2 \right] g_{n-m+1}(\beta_m, \dots, \beta_n) d\beta_m \dots d\beta_n \right) \wedge 1 \right) \xi_n(dv) \\ &= 2 \int_V \left(\sum_{k=m}^n \langle v, e_k \rangle^2 \wedge 1 \right) \xi_n(dv) \\ &= 2 \int_V \left(\sum_{k=m}^n \langle \pi_n v, e_k \rangle^2 \wedge 1 \right) (v_A + v_A^{-1})(dv) \\ &= 4 \int_V \left(\sum_{k=m}^n \langle v, e_k \rangle^2 \wedge 1 \right) v(dv). \end{aligned}$$

Condition (5.25) implies

$$\limsup_{m \rightarrow \infty} \sup_{n \geq m} \int_{\mathbb{R}^{n-m+1}} (1 - \text{Re } \varphi_{\vartheta_n}(\beta_m e_m + \dots + \beta_n e_n)) g_{n-m+1}(\beta_m, \dots, \beta_n) d\beta_m \dots d\beta_n = 0,$$

which shows by [25, Le.VI.2.3] that the family $\{\vartheta_n\}_{n \in \mathbb{N}}$ is relatively compact in $M(V)$. Since \tilde{Z}_A and thus its characteristic function $\varphi_{\tilde{Z}_A}$ are continuous (see [35, Pro.IV.3.4]), we have for each $v^* \in V^*$:

$$\lim_{n \rightarrow \infty} \varphi_{\vartheta_n}(v^*) = \lim_{n \rightarrow \infty} \varphi_{\tilde{Z}_A}(\langle e_1, v^* \rangle e_1 + \dots + \langle e_n, v^* \rangle e_n) = \varphi_{\tilde{Z}_A}(v^*). \quad (5.26)$$

Together with the relative compactness of $\{\vartheta_n\}_{n \in \mathbb{N}}$ it follows from Theorem IV.3.1 in [35] that the probability measures $\{\vartheta_n\}_{n \in \mathbb{N}}$ converges weakly to a measure ϑ , which coincides

with the cylindrical distribution of \tilde{Z}_A on $\mathcal{Z}(V)$. Consequently, [35, Th.IV.2.5] guarantees that there exists a random variable $\tilde{I}_A \in L_P^0(\Omega; V)$ with $\langle \tilde{I}_A, v^* \rangle = \tilde{Z}_A v^*$ for all $v^* \in V^*$. Thus, the cylindrical Lévy measure $\nu_A + \nu_A^-$ of \tilde{Z}_A extends to the Lévy measure of \tilde{I}_A . Since

$$\nu_A(C) \leq \nu_A(C) + \nu_A^-(C) \quad \text{for all } C \in \mathcal{Z}(V),$$

Theorem 3.4 implies that ν_A extends to a Lévy measure which shows Condition (5.10). \square

Remark 5.11. In the work [9] on Lévy processes in Hilbert spaces, the author develops among others a theory of stochastic integration for deterministic operators with respect to a classical Lévy process in a separable Hilbert space U . More specifically, let V be another separable Hilbert space and define the set

$$S := \left\{ f: [0, T] \rightarrow \mathcal{L}(U, V) : f \text{ is strongly measurable and } \int_0^T \|f(s)\|_{U \rightarrow V}^2 ds < \infty \right\}.$$

Then by using tightness conditions for infinitely divisible measures in Hilbert spaces (see [25]), a stochastic integral is defined for integrands in S in [9]. In this case of genuine Lévy processes, it is easy to see that each function $f \in S$ satisfies Condition (5.15) in Lemma 5.8 and Conditions (5.24) and (5.25) in Theorem 5.10. Thus, Theorem 5.10 guarantees that each $f \in S$ is stochastically integrable in our sense according to Definition 5.5.

Corollary 5.12. *Under the assumption of Theorem 5.6 let the cylindrical measure ν be defined by (5.10). Then we have the following:*

(a) *If V is of type $p \in [1, 2]$, then Condition (5.10) replaced by*

$$(3') \quad \nu \text{ extends to a measure and } \int_V (\|v\|^p \wedge 1) \nu(dv) < \infty,$$

implies together with (5.8) and (5.9) that f is stochastically integrable.

(b) *If V is of cotype $q \in [2, \infty)$ then stochastic integrability of f implies*

$$(3') \quad \nu \text{ extends to a measure and } \int_V (\|v\|^q \wedge 1) \nu(dv) < \infty.$$

Proof. The reformulation of Condition (5.10) in Theorem 5.6 follows in both cases from results in the article [10]. In this work the authors establish sufficient conditions in Banach spaces of type p and necessary conditions in Banach spaces of cotype q for a σ -finite measure to be a Lévy measure. \square

Corollary 5.13. *If V is the space $\ell^p(\mathbb{R})$ of sequences for $p \in [2, \infty)$ equipped with the standard basis $(e_k)_{k \in \mathbb{N}}$, then Condition (5.9) and (5.10) in Theorem 5.6 can be replaced by*

$$(2') \quad \sum_{k=1}^{\infty} \left(\int_0^T \langle f^*(s)e_k, Qf^*(s)e_k \rangle ds \right)^{p/2} < \infty. \quad (5.27)$$

(3') *the cylindrical measure ν extends to a Radon measure on $\mathcal{B}(V)$ and satisfies*

$$(a) \quad \int_V (\|v\|^p \wedge 1) \nu(dv) < \infty; \quad (5.28)$$

$$(b) \quad \sum_{k=1}^{\infty} \left(\int_{\|v\| \leq 1} |\langle v, e_k \rangle|^2 \nu(dv) \right)^{p/2} < \infty. \quad (5.29)$$

Proof. Theorem 5.6 in [35] guarantees that Conditions (5.9) and (5.27) are equivalent. The class of Lévy measures in $\ell^p(\mathbb{R})$ is described by Conditions (5.28) and (5.29) according to a result in [16]. \square

6 Ornstein-Uhlenbeck processes

In this last part we apply the previous developed theory of stochastic integration to define Ornstein-Uhlenbeck processes driven by cylindrical Lévy processes. These processes are important since they are solutions of stochastic evolution equations driven by cylindrical Lévy processes, see for instance [26]. We do not study this connection in this work but we consider examples of specific cases, then these processes exist.

If L is a cylindrical Lévy process in a Banach space U and $G \in \mathcal{L}(U, V)$ then the cylindrical Lévy process GL defined by $(GL)(t)v^* := L(t)(G^*v^*)$ for all $v^* \in V^*$ and $t \geq 0$ is a cylindrical Lévy process in V . It follows for a function $f: [0, T] \rightarrow \mathcal{L}(U, V)$ that if $f(\cdot) \circ G$ is stochastically integrable w.r.t. L then f is stochastically integrable w.r.t. GL and

$$\int_0^T f(s)G dL(s) = \int_0^T f(s) d(GL)(s).$$

Thus, without loss of generality we can assume in this section $U = V$, i.e. L is a cylindrical Lévy process in the separable Banach space V .

Definition 6.1. If a strongly continuous semigroup $(T(t))_{t \in [0, T]}$ on V is stochastically integrable with respect to L , then we call the stochastic process $(X(t) : t \in [0, T])$ defined by

$$X(t) := T(t)v_0 + \int_0^t T(t-s) dL(s) \quad \text{for all } t \in [0, T],$$

Ornstein-Uhlenbeck process with initial value $v_0 \in V$ driven by L .

The existence of the stochastic convolution integral in Definition 6.1 is guaranteed by the following result.

Lemma 6.2. *A function $f: [0, T] \rightarrow \mathcal{L}(V, V)$ is stochastically integrable if and only if $f(T-\cdot)$ is stochastically integrable. In this case we have the equality in distribution:*

$$\int_0^T f(s) dL(s) \stackrel{d}{=} \int_0^T f(T-s) dL(s).$$

Proof. If f is weakly in $\mathcal{G}([0, T]; V^*)$ then $f(T-\cdot)$ is also weakly in $\mathcal{G}([0, T]; V^*)$. Since Conditions (5.8) - (5.10) in Theorem 5.6 are invariant under a transformation $s \mapsto T-s$ the first part of the Lemma is proved. The identity

$$\exp \left(\int_0^T \Psi(f^*(s)v^*) ds \right) = \exp \left(\int_0^T \Psi(f^*(T-s)v^*) ds \right) \quad \text{for all } v^* \in V^*,$$

shows the equality of the distributions by Lemma 5.4 \square

As an example of an Ornstein-Uhlenbeck process we consider the case of a diagonalisable semigroup and of a cylindrical Lévy process defined by a sum acting independently along the eigenbasis of the semigroup, cf. Lemma 4.2. This kind of setting is considered in several publications, e.g. [27] and [28].

Corollary 6.3. Assume that V is a Hilbert space and that there exists an orthonormal basis $(e_k)_{k \in \mathbb{N}}$ of V and $(\gamma_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}$ such that the semigroup $(T(t))_{t \in [0, T]}$ satisfies

$$T^*(t)e_k = e^{\gamma_k t} e_k \quad \text{for all } t \in [0, T], k \in \mathbb{N}. \quad (6.1)$$

Let the cylindrical Lévy process L be of the form

$$L(t)v^* = \sum_{k=1}^{\infty} \langle e_k, v^* \rangle \ell_k(t) \quad \text{for all } v^* \in V^*, t \geq 0,$$

where $(\ell_k)_{k \in \mathbb{N}}$ is a sequence of independent, symmetric Lévy processes in \mathbb{R} with characteristics $(0, 0, \nu_k)$. Then the semigroup $(T(t))_{t \in [0, T]}$ is stochastically integrable w.r.t. L if and only if

$$\sum_{k=1}^{\infty} \int_0^T \int_{\mathbb{R}} \left(e^{2\lambda_k s} |\beta|^2 \wedge 1 \right) \nu_k(d\beta) ds < \infty. \quad (6.2)$$

Proof. According to Lemma 4.2, the cylindrical Lévy process L has characteristics $(0, 0, \mu)$ satisfying $(\mu \circ e_k^{-1})(d\beta) = \nu_k(d\beta)$ for all $k \in \mathbb{N}$. Independence of the Lévy processes $(\ell_k)_{k \in \mathbb{N}}$ implies for the Lévy measure $\mu \circ \pi_{e_m, \dots, e_n}^{-1}$ of $(L(1)(e_m), \dots, L(1)(e_n))$ for $0 \leq m \leq n$ the identity

$$\mu \circ \pi_{e_m, \dots, e_n}^{-1} = \sum_{k=m}^n \underbrace{\delta_0 \otimes \dots \otimes \delta_0}_{k-m \text{ times}} \otimes (\mu \circ \pi_{e_k}^{-1}) \otimes \underbrace{\delta_0 \otimes \dots \otimes \delta_0}_{n-k \text{ times}}.$$

Thus, we obtain for all $v \in V$:

$$\begin{aligned} & \int_0^T \int_V \left(\sum_{k=m}^n \langle v, T^*(s)e_k \rangle^2 \wedge 1 \right) \mu(dv) ds \\ &= \int_0^T \int_V \left(\sum_{k=m}^n \langle v, e^{\lambda_k s} e_k \rangle^2 \wedge 1 \right) \mu(dv) ds \\ &= \int_0^T \int_{\mathbb{R}^{n-m+1}} \left(\sum_{k=m}^n |e^{\lambda_k s} \beta_k|^2 \wedge 1 \right) (\mu \circ \pi_{e_m, \dots, e_n}^{-1})(d\beta_m \dots d\beta_n) ds \\ &= \sum_{k=m}^n \int_0^T \int_{\mathbb{R}} (e^{2\lambda_k s} \beta^2 \wedge 1) (\mu \circ e_k^{-1})(d\beta) ds. \end{aligned}$$

Since V is a Hilbert space, the adjoint semigroup $(T^*(t))_{t \in [0, T]}$ is strongly continuous, and thus $(T(t))_{t \in [0, T]}$ is weakly in $\mathcal{G}([0, T]; V^*)$. An application of Theorem 5.10 establishes that the semigroup is stochastically integrable if and only if (6.2) is satisfied. \square

Example 6.4. Assume that the cylindrical Lévy process L is given as in Example 4.5 by $\ell_k(\cdot) := \sigma_k h_k(\cdot)$, where $(h_k)_{k \in \mathbb{N}}$ is a sequence of independent, symmetric, α -stable processes and $(\sigma_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}$. If the strongly continuous semigroup $(T(t))_{t \in [0, T]}$ satisfies (6.1) for $\lambda_k < 0$ with $\lambda_k \rightarrow -\infty$ for $k \rightarrow \infty$, then a simple calculation shows that (6.2) is satisfied if and only if

$$\sum_{k=1}^{\infty} \frac{|\sigma_k|^\alpha}{|\gamma_k|} < \infty.$$

In this case, the semigroup is stochastically integrable, which coincide with a result in [27].

We now consider stochastic integrability of a semigroup $(T(t))_{t \geq 0}$ w.r.t. to a cylindrical Lévy noise constructed by subordination as in Example 4.7. In fact, we will show integrability in a possible smaller subspace $E \subseteq V$ with norm $\|\cdot\|_E$ assuming $T(t)(V) \subseteq E$ for almost all $t > 0$. Recall that H_C denotes the reproducing kernel Hilbert space of the subordinated cylindrical Wiener process and $i_C : H_C \rightarrow V$ its embedding. In the following denote by $R(H_C, E)$ the space of γ -radonifying operators $g : H_C \rightarrow E$. For $g \in R(H_C, E)$ and $p \in [1, \infty)$ define

$$\|g\|_{R_p(H_C, E)}^p := E \left\| \sum_{k=1}^{\infty} \gamma_k g h_k \right\|_E^p,$$

where $(\gamma_k)_{k \in \mathbb{N}}$ is a family of independent, real valued standard normally distributed random variables and $(h_k)_{k \in \mathbb{N}}$ is an orthonormal basis in H_C .

Corollary 6.5. *Let L be the cylindrical Lévy process in a separable Banach space V defined by*

$$L(t)v^* := W(\ell(t))v^* \quad \text{for all } v^* \in V^*, t \geq 0, \quad (6.3)$$

where W denotes a cylindrical Wiener process with covariance operator C and ℓ is a real valued subordinator with characteristics $(0, 0, \rho)$. Let E be a separable Banach space of type $p \in [1, 2]$. If the semigroup $(T(t))_{t \in [0, T]}$ in V is weakly in $\mathcal{G}([0, T]; V^*)$ and the mapping $T(t) \circ i_C$ is in $R(H_C, E)$ for almost all $t \in [0, T]$, then

$$\int_0^\infty \int_0^T \left(|r|^{p/2} \|T(s) \circ i_C\|_{R_p(H_C, E)}^p \wedge 1 \right) ds \rho(dr) < \infty,$$

implies that the semigroup $(T(t))_{t \in [0, T]}$ is stochastically integrable w.r.t. L and the stochastic integral is E -valued.

Proof. Let γ denote the canonical cylindrical Gaussian measure on H_C and let κ be defined as in Lemma 4.8, i.e. $\kappa(h, s) := \sqrt{s} i_C h$, and χ as in Theorem 5.6, i.e. $\chi(s, v) := T(s)v$. For applying Corollary 5.12 we have to show that the cylindrical measure $\nu = ((\gamma \otimes \rho) \circ \kappa^{-1}) \otimes \text{leb} \circ \chi^{-1}$ extends to a measure on $\mathcal{B}(V)$. For this purpose define the family of cylindrical sets

$$G := \{C(v_1^*, \dots, v_n^*; R) : v_1^*, \dots, v_n^* \in V^\odot, \\ R = (a_1, b_1) \times \dots \times (a_n, b_n), -\infty \leq a_j < b_j \leq \infty, j = 1, \dots, n, n \in \mathbb{N}\},$$

where V^\odot denotes the weak* dense subspace of V^* such that $(T^*(t))_{t \in [0, T]}$ acts strongly continuously on V^\odot . Since V is separable and V^\odot separates points in V , Theorem I.2.1 in [35] guarantees that G generates the σ -algebra $\mathcal{B}(V)$. Define $\gamma_s := \gamma \circ (T(s) \circ i_C)^{-1}$ for all $s \in [0, T]$. Let $\Gamma : H_C \rightarrow L_p^0(\Omega; \mathbb{R})$ be a cylindrical random variable with distribution γ . Since $\Gamma(T^*(s_k)v^*) \rightarrow \Gamma(T^*(s)v^*)$ in probability for $s_k \rightarrow s$ and all $v^* \in V^\odot$, the portmanteau theorem in \mathbb{R}^n implies for each $C := C(v_1^*, \dots, v_n^*; R) \in G$:

$$\begin{aligned} \gamma_{s_k}(C) &= P((\Gamma(T^*(s_k)v_1^*), \dots, \Gamma(T^*(s_k)v_n^*)) \in R) \\ &\rightarrow P((\Gamma(T^*(s)v_1^*), \dots, \Gamma(T^*(s)v_n^*)) \in R) = \gamma_s(C) \quad \text{as } s_k \rightarrow s. \end{aligned} \quad (6.4)$$

Consequently, Theorem 452C in [13] on disintegration of measures implies that $s \mapsto \gamma_s(B)$ is measurable for all $B \in \mathcal{B}(V)$ and almost all $s \in [0, T]$, and that there exists a Borel measure

ϑ on $\mathcal{B}(V)$ such that

$$\vartheta(B) = \int_0^T \gamma_s(B) ds \quad \text{for all } B \in \mathcal{B}(V). \quad (6.5)$$

Define for each $r \in (0, \infty)$ the measure $\vartheta_r(B) := \vartheta(r^{-1/2}B)$ for all $B \in \mathcal{B}(V)$. If a sequence $(r_k)_{k \in \mathbb{N}} \subseteq (0, \infty)$ converges to $r \in (0, \infty)$ then it follows similarly as in (6.4) that $\gamma_s(r_k^{-1/2}C) \rightarrow \gamma_s(r^{-1/2}C)$ for all $s \in [0, T]$ and $C \in G$. By applying Lebesgue's theorem of dominated convergence we conclude from (6.5) that the mapping $r \mapsto \vartheta_r(C)$ is continuous for all $C \in G$. Another application of Theorem 452C in [13] implies that there exists a measure μ on $\mathcal{B}(V)$ satisfying

$$\mu(B) = \int_0^\infty \vartheta_r(B) \rho(dr) = \int_0^\infty \vartheta(r^{-1/2}B) \rho(dr) \quad \text{for all } B \in \mathcal{B}(V).$$

Note that by our argument above the function $(s, r) \mapsto \gamma_s(r^{-1/2}C)$ is separately continuous in both variables for all $C \in G$ and thus jointly measurable. Tonelli's theorem enables us to conclude for all $C \in G$ that

$$\mu(C) = \int_0^\infty \int_0^T \gamma_s(r^{-1/2}C) ds \rho(dr) = \int_0^T \int_0^\infty \gamma_s(r^{-1/2}C) \rho(dr) ds = \nu(C),$$

which shows that ν extends to the measure μ on $\mathcal{B}(V)$. The measure μ satisfies

$$\begin{aligned} \int_V \left(\|v\|^p \wedge 1 \right) \mu(dv) &= \int_0^T \int_0^\infty \int_{H_C} \left(\|T(s)(\sqrt{r} i_C h)\|^p \wedge 1 \right) \gamma(dh) \rho(dr) ds \\ &\leq \int_0^T \int_0^\infty \left(|r|^{p/2} \int_{H_C} \left(\|T(s)(i_C h)\|^p \right) \gamma(dh) \wedge 1 \right) \rho(dr) ds \\ &= \int_0^T \int_0^\infty \left(|r|^{p/2} \|T(s) \circ i_C\|_{R_p(H_C, E)}^p \wedge 1 \right) \rho(dr) ds. \end{aligned}$$

An application of Corollary 5.12 completes the proof. \square

A very similar result as Corollary 6.5 is derived in [8]. However, the conditions in our result are purely intrinsic, whereas the result in [8] is based on conditions in terms of an additional Banach space, which is not related to the problem under consideration.

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